Completely faithful modules and
quasi-Frobenius algebras

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Introduction.

A generator in the category (cf. [12]) of left (right) modules over a ring Λ will be called a completely faithful left (right) Λ-module according to [4]. The complete faithfulness of modules is a Morita invariant property, which plays an essential part for the categorical theory of rings.

B. Müller introduced in [19] the notion of a quasi-Frobeninus extension Λ of a ring Ω. In case Ω is in the center of Λ, i.e., in case Λ is a Ω-algebra, this coincides with that of a semi-Frobenius algebra in [15], and, in this paper, we shall call this a quasi-Frobenius algebra. An algebra Λ over a commutative ring R, which is a finitely generated projective R-module, will be called a quasi-Frobenius R-algebra, if $\Lambda^* = \mathrm{Hom}_R(\Lambda, R)$ is a completely faithful left (and right) Λ-module. The purpose of this paper is to show some basic properties in quasi-Frobenius algebras.

As is well known, any completely faithful Λ-module is faithful, but a faithful Λ-module is not always completely faithful, and if Λ is commutative, then any finitely generated, faithful, projective Λ-module is completely faithful. It is also known (cf. [3] or [9]) that, in case Λ is a quasi-Frobenius ring, any faithful Λ-module is completely faithful, and it was proved in [1] that, if Λ is a maximal order over a Dedekind domain in a central simple algebra, then any finitely generated projective Λ-module is completely faithful. However, it seems that such facts have not been treated systematically. Recently, G. Azumaya gave in [4] a characterization of a ring Λ with the property: (G) Any faithful Λ-module is completely faithful. Another purpose of this paper is to examine the structure of a ring Λ with each of the following properties:

(FG) Any finitely generated, faithful Λ-module is completely faithful.

(PFG) Any finitely generated, faithful, projective Λ-module is completely faithful.