J. Math. Soc. Japan
Vol. 19, No. 3, 1967

Projective and injective limits of weakly compact sequences of locally convex spaces

By Hikosaburo KOMATSU

(Received Feb. 14, 1967)

Silva [15] and Raikov [12] [13] studied projective and injective limits of compact sequences of locally convex spaces and revealed remarkable properties of the locally convex spaces expressed as those limits. However, they do not seem to have noticed at first that those spaces are exactly the Fréchet Schwartz spaces and their strong dual spaces discussed by Grothendieck [5].

We extend their results to the limit spaces of weakly compact sequences of locally convex spaces and show that almost all important properties are preserved. We presuppose only the text of Bourbaki [1] except for the closed range theorem and the definition of (DF) spaces.

A projective (injective) sequence of locally convex spaces with (one-one) continuous linear mappings:

$$X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_n \longleftarrow \cdots$$

$$(X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n \longrightarrow \cdots)$$

is said to be weakly compact or compact if all mappings are weakly compact or compact respectively. The limit space $\varprojlim X_j$ ($\varinjlim X_j$) of a weakly compact or compact projective (injective) sequence is said to be (FS*) or (FS) ((DFS*) or (DFS)) respectively. (FS*) spaces are totally reflexive and Fréchet and (FS) spaces are also separable and Montel. (DFS*) spaces are Hausdorff, totally reflexive, fully complete, bornologic and (DF), and (DFS) spaces are moreover separable and Montel.

Closed subspaces, quotient spaces and projective limits of sequences of (FS*) spaces ((FS) spaces) are (FS*) ((FS)). Closed subspaces, quotient spaces and injective limits of sequences of (DFS) spaces are (DFS). Quotient spaces and direct sums of sequences of (DFS*) spaces are (DFS*). Closed subspaces of (DFS*) spaces are not always (DFS*). However, the bornologic topology and the Mackey topology associated with the induced topology are the same on any closed subspace and they make the subspace into a (DFS*) space.

The strong dual spaces of (FS*) spaces ((FS) spaces) are (DFS*) ((DFS)) and conversely the strong dual spaces of (DFS*) spaces ((DFS) spaces) are