J. Math. Soc. Japan
Vol. 18, No. 4, 1966

# Remarks on evolution inequalities

By J. L. LIONS[1)]

(Received May 17, 1966)

## Introduction

The Lax-Milgram Lemma was extended by G. Stampacchia [4] to the following: if $a(u, v)$ is a continuous bilinear form *coercive* on a (real) Hilbert space $V$ and if $K$ is a closed convex set in $V$, then, given a continuous linear form $v \rightarrow L(v)$ on $V$, there exists a unique element $u$ in $K$ such that $a(u, v-u) \geqq L(v-u)$ $\forall v \in K$ (the Lax-Milgram Lemma corresponds to the case when $K=V$).

In a joint work with Stampacchia [**10**] [**11**] we studied similar problems for bilinear forms which are (i) either $\geqq 0$ but *not coercive* (ii) either defined *on two different Hilbert spaces.*

In this paper we give some complements to the result of [**11**] on (ii). This will solve, as a particular case—see Section 3.3. below—the following *non linear boundary value problem*: find a function $u(x, t)$, $x \in \Omega \subset \boldsymbol{R}^n$, $t \in (0, T)$, $T < \infty$, such that:

$$\frac{\partial u}{\partial t} - \Delta u + u = f \qquad \left(\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right), \tag{1}$$

$$u \geqq 0 \quad \text{on} \quad \Gamma \times (0, T) \qquad (\Gamma = \text{boundary of } \Omega) \tag{2}$$

$$\frac{\partial u}{\partial \nu} \geqq 0 \quad \text{on} \quad \Gamma \times (0, T) \qquad \left(\frac{\partial}{\partial \nu} = \text{exterior normal derivative}\right)$$

$$u \cdot \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times (0, T),$$

$$u(x, 0) = u(x, T). \tag{3}$$

(The case when instead of the "*periodic problem*" (3) we consider the "*initial value problem*" $u(x, 0) =$ given was solved in [**11**].)

Section 1 gives a general existence theorem; Section 2 gives applications to "ordinary" evolution equations and Section 3 gives a general existence

1) This paper develops technical details of part of a lecture given at the Annual Meeting of the Math. Soc. Japan, Kyoto, May 1966. Other parts of the lecture corresponded to a joint work with G. Stampacchia [**10**] [**11**].