J. Math. Soc. Japan
Vol. 17, No. 4, 1965

On Kronecker's limit formula in a totally imaginary quadratic field over a totally real algebraic number field

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(Received Feb. 15, 1965)

Introduction

Let K be an algebraic number field of finite degree and $\zeta_K(s)$ be the Dedekind zeta-function of K. Then $\zeta_K(s)$ has an expansion of the form

$$\zeta_K(s)=A_{-1}/(s-1)+A_0+A_1(s-1)+\cdots.$$

Here A_{-1}, the residue of $\zeta_K(s)$ at $s=1$, was determined by Dirichlet and Dedekind for any algebraic number field K. However, little is known about the constant term A_0, in spite of its importance. As far as the author knows, A_0 has been investigated only in the cases where K is either a cyclotomic field or a quadratic field. The determination of A_0 for imaginary quadratic fields is known as "Kronecker's limit formula". The purpose of this paper is to consider this problem for any totally imaginary quadratic extension K of a totally real algebraic number field k. The main results are as follows. For any absolute ideal class $\mathfrak{K}$ in K, let $\zeta_K(s;\mathfrak{K})$ denote the zeta-function of the class $\mathfrak{K}$. Let n be the degree of k. We shall show that, in the expansion

$$\zeta_K(s;\mathfrak{K})=a_{-1}/(s-1)+a_0+a_1(s-1)+\cdots,$$

the constant a_0 can be expressed as a special value of $\log \Psi_k(z^{(1)},\cdots z^{(n)};\mathfrak{m},\mathfrak{n})$ with a certain analytic function Ψ_k defined on the product of n-copies of complex upper half-planes (Theorem 1.2). This function is a generalization of Dedekind's η-function. But this function cannot be a Hilbert's modular form. This fact was kindly mentioned to me by Professor Siegel. As an application we shall obtain a formula for the quotient of the class number of the absolute class field over K divided by the class number of K (Theorem 3).

Here the author wishes to express his hearty thanks to Professor C. L. Siegel.