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On the field of definition of Borel subgroups of semi-simple algebraic groups

Dedicated to Professor Y. Akizuki for his 60th birthday

By Takashi ONO

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Let k be a perfect field and let G be a connected semi-simple algebraic group defined over k. It is known that G has a maximal torus T defined over k (Rosenlicht [2]). Fixing once for all such a torus T, denote by $\boldsymbol{B}$ the set of all Borel subgroups of G containing T. Our purpose is to prove the following

THEOREM. *Every group in $\boldsymbol{B}$ is defined over k if and only if T is trivial over k. When that is so, all groups in $\boldsymbol{B}$ are conjugate by k-rational points of the normalizer of T.*

For some purpose the following trivial restatement is useful.

COROLLARY. *Let K/k be an extension such that K is perfect. Then, every group in $\boldsymbol{B}$ is defined over K if and only if T is split by K. When that is so, all groups in $\boldsymbol{B}$ are conjugate by K-rational points of the normalizer of T.*

PROOF OF THEOREM. We begin with arranging the basic notions in Séminaire Chevally [1] from the Galois theoretical view point.

Denote by N the normalizer of T and by W the Weyl group N/T of T. Let $\bar{k}$ be the algebraic closure of k and $\mathfrak{g}=\mathfrak{g}(\bar{k}/k)$ be the Galois group of $\bar{k}/k$. Since every coset of W contains a $\bar{k}$-rational point, one can define the action of $\mathfrak{g}$ on W by

$$w^{\sigma}=s^{\sigma} \bmod T, \quad \text{where} \quad w=s \bmod T \quad \text{and} \quad s\in N_{\bar{k}}.^{*}$$

The group $\mathfrak{g}$ acts on the character module $\hat{T}$ since every character is $\bar{k}$-rational. Furthermore, W acts on $\hat{T}$ by

$$(w\chi)(t)=\chi(s^{-1}ts), \quad \text{where} \quad w=s \bmod T, \quad s\in N.$$

One verifies easily that

$$(w\chi)^{\sigma}=w^{\sigma}\chi^{\sigma} \qquad \text{for} \quad \sigma\in\mathfrak{g},\ w\in W,\ \chi\in\hat{T}.$$

In other words, $\hat{T}$ has a $(\mathfrak{g}, W)$-module structure. By linearity, this structure is trivially extended to the vector space $\hat{T}^{\boldsymbol{Q}}=\boldsymbol{Q}\otimes\hat{T}$.

* For an algebraic set A we denote by A_K the subset of K-rational points.