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Results on the order of holomorphic functions defined in the unit disk[1)]

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§ 1. Introduction

1. Let D denote the open unit disk and C the unit circle in the complex plane. Further, let $S(e^{i\theta}, \alpha)$ denote the symmetric Stolz domain at $e^{i\theta}$ of opening 2α lying in D. For certain classes of functions, holomorphic in D, results are known concerning the order, or growth, of functions belonging to the class. These results usually take two forms. One group of theorems gives a type of global order. For example, if $f(z)$ is univalent and holomorphic in D then Koebe's distortion theorem gives that $|f'(z)| \leqq \frac{(1+|z|)}{(1-|z|)^3}$. However if we restrict the choice of z somewhat a better estimate on the order can be given. Seidel and Walsh ([**15**], p. 338) showed that $|f'(z)|(1-|z|)^{\frac{1}{2}} \to 0$ as z tends to $e^{i\theta}$, $z \in S(e^{i\theta}, \alpha)$, for any $\alpha > 0$ and almost all $\theta \in [0, 2\pi)$. This type of result has been called a "statistical" result on order by J. Lelong-Ferrand.

If $P(z)$ is any function, holomorphic in D, which omits in D the values 0 and 1, then, as is well known, Schottky's theorem gives a global order for $P(z)$ to the effect that $|P(z)| \leqq e^{\frac{A}{(1-|z|)}}$ where A is a positive constant depending on $P(0)$. The main result in this paper will be to give a statistical theorem concerning the order of $P(z)$. In its simplest form the theorem states that for almost all $\theta \in [0, 2\pi)$, any fixed $\mu > 0$ and $\varepsilon > 0$, $|P(z)|\, e^{\frac{-\mu}{(1-|z|)^{1/2+\varepsilon}}}$ tends to 0 as z tends to $e^{i\theta}$ in any Stolz domain at $e^{i\theta}$. Thus, as in the case of univalent functions, a smaller estimate can be given for almost all $\theta \in [0, 2\pi)$, on sequences approaching $e^{i\theta}$ within any Stolz domain at $e^{i\theta}$.

In § 2 we deduce the fundamental theorem used to prove the main theorem. This fundamental theorem is similar in content to a result of Lelong-Ferrand ([**10**], p. 23).

Some results are given in § 3 on the order of functions holomorphic in D for which information is known concerning the order of their Taylor coeffi-

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