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The index of coset spaces of compact Lie groups

By Akio HATTORI

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§ 1. Introduction.

Let G be a compact connected Lie group and H a closed connected subgroup of G. We shall denote by $r(G)$ and $r(H)$ the ranks of G and H respectively. In the present note, we shall prove that, if $r(H) < r(G)$, then the index $\tau(G/H)$ of G/H (in the sense of Thom) vanishes; and that, if $r(H) = r(G)$, then the index can be expressed as the integral of some central function on H over the group manifold H. The precise statement will be given by Theorem 1 which we shall obtain at the end of § 3.

In the latter case, Borel-Hirzebruch [1] gave a formula which expresses the index $\tau(G/H)$ in terms of roots of G and those of H. They computed actually the L-genus which, as the index theorem of Thom-Hirzebruch asserts, coincides with the index. In § 4 we evaluate the integral in Theorem 1 to derive the formula of Borel-Hirzebruch. Here we do not make use of the index theorem. Thus our result can be regarded as providing a new proof of the index theorem for the space G/H with $r(H) = r(G)$.

§ 2. The index $\tau(G/H)$.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to the analytic subgroup H. There exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ which is complementary to $\mathfrak{h}$ and is invariant under the adjoint representation of H. We shall denote by Λ the exterior algebra of $\mathfrak{m}$ and by Λ^* the exterior algebra of the dual $\mathfrak{m}^*$ of the vector space $\mathfrak{m}$. The adjoint representation of H on $\mathfrak{m}$ extends to representations of H on Λ and on Λ^* in the standard fashion. Let us denote by Λ^H and Λ^{*H} the subalgebras of Λ and Λ^* respectively consisting of elements fixed under all operations of H. The algebra Λ^{*H} may be canonically identified with the algebra of G-invariant differential forms on G/H, and, as such, carries a differential operator d. The real cohomology ring $H^*(G/H, \boldsymbol{R})$ is then the derived ring of Λ^{*H} with respect to d.

Let e be a non zero element of Λ^n where Λ^n denotes the n-th exterior product of $\mathfrak{m}$, and $n = \dim \mathfrak{m} = \dim G/H$. Since H is compact and connected, e