J. Math. Soc. Japan
Vol. 13, No. 3, 1961

On the zeta-functions of the algebraic curves uniformized by certain automorphic functions

By Goro SHIMURA

(Received March 31, 1961)

Introduction. After Hasse and Weil, we can attach a zeta-function to every algebraic variety defined over an algebraic number field. In contrast with its importance, our knowledge of the zeta-function of this kind is little. At present, as far as I know, the zeta-function is determined only in the following two cases.

I) Abelian varieties with sufficiently many complex multiplications [**30**, **3**, **27**].

II) Algebraic curves uniformized by modular functions belonging to congruence-subgroups [**6**, **22**].

Here we note that the determination of the zeta-function of a curve is essentially the same as the determination of the zeta-function of its jacobian. Now, in all these cases, the zeta-functions are meromorphic on the whole complex plane and satisfy functional equations, as conjectured by Hasse.

The purpose of the present paper is to supply a new class of algebraic curves, for which Hasse's conjecture is true, and of which the curves of II) are particular cases. Our principal result is as follows. Let Φ be an indefinite quaternion algebra over the rational number field $\boldsymbol{Q}$, and $\mathfrak{o}$ a maximal order in Φ. Take a positive integer N which is prime to the discriminant of Φ and denote by Γ_N the group of units γ of $\mathfrak{o}$, with positive reduced norm, such that $\gamma \equiv 1 \bmod. N\mathfrak{o}$. As Φ has a faithful representation by real matrices of degree 2, Γ_N is considered as a Fuchsian group on the upper half plane $\mathfrak{H}$. If Φ has no zero-divisor, $\Gamma_N \backslash \mathfrak{H}$ is compact, while if Φ is the total matric algebra of degree 2 over $\boldsymbol{Q}$, Γ_N is nothing but the principal congruence-subgroup of $\mathrm{SL}(2, \boldsymbol{Z})$ of level N. Now, according to Eichler [**7**], we can develop the theory of Hecke's operators for cusp-forms with respect to Γ_N. We obtain then Dirichlet-series $D(s)$, meromorphic on the whole plane, having Euler-products, and satisfying functional equations. Let $\mathfrak{K}_N$ be the field of automorphic functions with respect to Γ_N. We can find an algebraic curve $\mathfrak{C}_N$, defined over $\boldsymbol{Q}$, whose function-field is identified with $\mathfrak{K}_N$. Our main theorem asserts that the zeta-function of $\mathfrak{C}_N$ is determined by the Dirichlet-series $D(s)$ for cusp-forms of degree 2.