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The representation of finite groups in algebraic number fields

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1. Introduction

Let $\mathfrak{G}$ be a finite group. We may state our problem naively as follows. Suppose given an absolutely irreducible representation of $\mathfrak{G}$. It has been known since the time of Frobenius that the given representation may be replaced by an equivalent representation in which all the coefficients lie in a field of algebraic numbers of finite degree over the rational field. The problem is to decide which number fields may be used and in particular to determine how small a field will suffice. The full answer to the question is not yet known. If $\mathfrak{G}$ has exponent n, then it is plausible that the field of n-th roots of unity will suffice for all the absolutely irreducible representations of $\mathfrak{G}$. This quite plausible conjecture was made by Maschke around 1900, but the first proof was given by R. Brauer just a few years ago.

In order to study the problem, Schur introduced a numerical invariant which has come to be known as the Schur index of the representation. This is, roughly speaking, a measure of the size of the smallest field that will do. Since the time of Schur, his invariant has been given a more natural significance in a more general setting as part of the theory of algebras. However, even though there exist several characterizations of the index, none of them will serve to determine it in terms of the table of characters and the multiplication table for the group.

The present paper is a small contribution toward this problem. Section 2 contains some preliminary definitions and remarks. In Section 3 we prove a lemma which is used in Section 4 to prove two theorems on the structure of certain rings of characters associated with a finite group $\mathfrak{G}$ and an algebraic number field K. The first of these is a new proof of a theorem of Witt, using a method due to Brauer and Tate. The second is an easy corollary. The same techniques will prove a theorem of Brauer which reduces the problem of determination of the indices for the representations of $\mathfrak{G}$, to the problem of determination of the indices for the representations of certain solvable sub-