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On the Goldbach problem in an algebraic number field I.

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§ 1. Introduction.

The famous but yet unsolved problem of Goldbach is to decide whether the following conjecture is true: every even positive rational integer except 2 and 4 will be represented as the sum of two odd prime numbers.

Concerning this problem, Vinogradov [7] proved in 1937 that every large odd integer is represented as the sum of three prime numbers, and obtained also an asymptotic formula for the number of representations. Estermann [1] proved then, in 1938, using the result of [7], that almost all even rational integers are represented as the sum of two prime numbers.

The purpose of this paper is to generalize these results to the case of algebraic number fields. Our final results will be stated as Theorem 10.1 and Theorem 11.1 in § 10 and § 11 respectively, but we shall give here an outline of our results.

Let K be an algebraic number field of degree n. This and the following notations will be used throughout this paper.

$K^{(1)}, K^{(2)}, \cdots, K^{(r_1)}$ are the real conjugates of K; $K^{(r_1+1)}, \cdots, K^{(r_1+r_2)}$, $K^{(r_1+r_2+1)} = \bar{K}^{(r_1+1)}, \cdots, K^{(n)} = \bar{K}^{(r_1+r_2)}$ are the complex conjugates of K.

We denote by $\mathfrak{o}$ the ideal consisting of all integers of K, by $\mathfrak{d}$ the *differente* of K and by $D = N(\mathfrak{d})$ (norm of $\mathfrak{d}$) the absolute value of the *discriminant* of K.

Let γ be a number of K and put $\mathfrak{d}\gamma = \mathfrak{b}/\mathfrak{a}$ with integral ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $(\mathfrak{a}, \mathfrak{b}) = 1$. We call $\mathfrak{a}$ the *denominator* of γ and denote this relation by $\gamma \rightarrow \mathfrak{a}$.

If μ is a number of K, we have an n-dimensional complex vector $(\mu^{(1)}, \mu^{(2)}, \cdots, \mu^{(n)})$ with real $\mu^{(q)}$ $(q = 1, 2, \cdots, r_1)$ and complex $\mu^{(p+r_2)} = \bar{\mu}^{(p)}$ $(p = r_1+1, \cdots, r_1+r_2)$, where $\mu^{(i)}$ is the conjugate of μ in $K^{(i)}$ $(i = 1, 2, \cdots, n)$. We shall denote this vector also by μ. We shall consider more generally any n-dimensional complex vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ with real $\xi_1, \cdots, \xi_{r_1}$ and complex $\xi_{p+r_2} = \bar{\xi}_p$ $(p = r_1+1, \cdots, r_1+r_2)$. For such ξ, we write

$$S(\xi) = \sum_{j=1}^{n} \xi_j, \qquad N(\xi) = \prod_{i=1}^{n} \xi_i$$

and put