Journal of the Mathematical Society of Japan Vol. 12, No. 2, April, 1960

On the recursive functions of ordinal numbers.

Dedicated to Professor Z. Suetuna for his 60th birthday.

By

Gaisi TAKEUTI

(Received May 26, 1959)

In a former paper [6], the author developed a theory of ordinal numbers independently of the set theory and then constructed the set theory in the theory of ordinal numbers.

In that theory, we used only the predicates $<$, $=$ and only the functions N, max, Iq, j, Min, Rec, and χ. (We used other special variables and functions 0, ω, δ, represented by the above described functions.)

In this paper, we shall call a function *semi-recursive* if it is represented by N, max, Iq, j, Min and Rec, and a semi-recursive function *recursive*, if every Min in the function satisfies the well-known condition as in the case of the recursive functions of natural numbers (to be given precisely later). We shall define, moreover, $\mathfrak{M}_a$ as the model generated by N, max, Iq, j, Min, Rec and the ordinals less than a. $\mathfrak{M}_a$ is well-ordered by the original order and has the same order type as the ordinal $\mathfrak{m}(a)$. Then we shall prove that an interpretation of a recursive function f in the model of ordinals less than $\mathfrak{m}(a)$ is f itself and that the power of $f(a_1, \cdots, a_n)$ is not greater than the power of $\max(a_1, \cdots, a_n)$, if f is recursive and $\max(a_1, \cdots, a_n) \geqq \omega$. It seems very difficult to generalize this proposition to the case of semi-recursive functions, because the consistency of the set theory could be proved, if it is proved.

On the formalized system developed in [6], we shall prove that there exists a recursive function C such that we can replace the axiom of cardinal by the weaker axiom $\forall x \exists y \forall z (C(x, y, z) = 0 \wedge y > 0)$ to construct the set theory. We shall give further the condition for the ordinal a with countable power that the ordinals less than a constitute the model of the set theory.

The author wishes to express his thanks to Prof. K. Gödel, who has given him valuable remarks. This work was done under Appointment supported by the International Cooperation Administration under the Visiting Research Scientists Program administered by the National Academy of Sciences of the United States of America.