Journal of the Mathematical Society of Japan Vol. 11, No. 1, January, 1959

On the radial order of a univalent function.

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(Received Jan. 12, 1958)
(Revised Sept. 12, 1958)

1. In a recent note in this Journal Gehring [**5**] has given a new proof of the following theorem of Denjoy [**1**] and Seidel and Walsh [**6**].*

THEOREM 1. *Suppose that $f(z)$ is regular and univalent in $|z|<1$. Then for almost all θ*

$$f'(z)=o((1-|z|)^{-\frac{1}{2}}) \tag{1.1}$$

uniformly as $z\rightarrow e^{i\theta}$ in each Stolz domain.

Gehring's proof, though short, is far from elementary, since it depends on a difficult maximal theorem of Hardy and Littlewood. In this note we give an alternative proof of Theorem 1 which is considerably more elementary.

2. We require a simple identity concerning Cesàro means. Let $f(z)=\sum c_{n}z^{n}$, and let $\tau_{n}^{\alpha}(\theta)$ denote the n-th Cesàro mean of order α of the sequence $nc_{n}e^{ni\theta}$. Then it is well known that for any α and for $|z|<1$

$$\frac{zf'(ze^{i\theta})}{(1-z)^{\alpha}}=\sum_{1}^{\infty}E_{n}^{\alpha}\tau_{n}^{\alpha}(\theta)z^{n}\,, \tag{2.1}$$

where (as usual)

$$E_{n}^{\alpha}=\binom{\alpha+n}{n}=\frac{(\alpha+1)(\cdots)(\alpha+n)}{n\,!} \qquad (n>0)\,.$$

3. Consider now the proof of the theorem. A familiar argument [**6**] allows us to assume that the image of $|z|<1$ under $\zeta=f(z)$ has finite area, or that

$$\int_{0}^{1}\int_{-\pi}^{\pi}|f'(\rho e^{i\theta})|^{2}\rho d\theta d\rho<\infty\,. \tag{3.1}$$

We show first that if f satisfies (3.1), and if $\alpha>1/2$, then the series $\sum|\tau_{n}^{\alpha}(\theta)|^{2}$ is convergent p. p. This is a particular case of a more general result (Flett [**4**]), but we give the proof for the sake of completeness.

Applying Parseval's theorem to the function (2.1) we obtain

$$\sum_{1}^{\infty}(E_{n}^{\alpha})^{2}|\tau_{n}^{\alpha}(\theta)|^{2}\rho^{2n}\leqq\frac{\rho}{2\pi}\int_{-\pi}^{\pi}\frac{|f'(\rho e^{i\theta+it})|^{2}}{|1-\rho e^{it}|^{2\alpha}}\,dt\,. \tag{3.2}$$

* Various generalizations of the theorem are known (see, for example, Ferrand [**3**]), but we do not consider these here.