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On conformal transformations of Riemannian spaces.

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Introduction. In this paper, we mean by a space always a connected Riemannian space (with a positive definite metric) of class C^{∞} and of dimension $n>3$. We denote such a space generally with M, and the conformal transformation group of M with $C(M)$, the isometry group of M with $I(M)$. If M' is another space and $C(M)\subset I(M')$, the study of $C(M)$ will be reduced to that of $I(M')$. We shall say for brevity that M is *conformally reducible* if we can find an M' with $C(M)\subset I(M')$. Several authors have shown that spaces satisfying some conditions on the Ricci tensor have the property $C(M)\subset I(M)$ or $C^{0}(M)\subset I(M)$, where $C^{0}(M)$ means the identity component of $C(M)$ (see [8]). In §1 of this paper, we shall show that $C(M)\subset I(M')$ holds for a submanifold M' of M whenever M is not conformally flat. Thus, spaces which are not conformally flat, are conformally reducible in the sense explained above. This is based on a theorem (theorem H) essentially due to Hlavatý. From this remark follow very easily the following known results for example:

I. If an n-dimensional Riemannian space M $(n>4)$ admits the group $C(M)$ of conformal transformations with dimensions greater than $n(n-1)/2+7$, then M is conformally flat (H. Hiramatu [1]).

II. Let ρ be the natural representation on the tangent space T_p of M at a point $p\in M$ of the group $C_p(M)$ of isotropy of $C(M)$ at p. If $\rho(C_p(M))$ is not contained in the rotation group of T_p with the induced metric, the conformal tensor vanishes at p (S. Ishihara and M. Obata, see [8] p. 277).

As another application of theorem H we shall prove that a Riemannian homogeneous space which is not conformally flat does not admit a conformal transformation but for an isometry.

In §§2–4, we shall consider conformally flat spaces, to which the "method of reduction" is not applicable. If we denote the n-Möbius group (for definition see below) by $M(n)$, the contents of §2 may be summarized by the following three statements:

(1) We have always $C(M)\subset M(n)$ locally,

(2) $C(M)=M(n)$ locally if and only if M is conformally equivalent with S^n (S^n means the n-sphere, cf. [4]).

(3) If $C(M)$ is transitive, then there exists an open submanifold M'