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An operator-theoretical integration of the wave equation.

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§ I. Introduction and the theorem.

We consider the Cauchy problem for the wave equation in m-dimensional euclidean space E^m:

$$\frac{\partial^2 u(t,x)}{\partial t^2}=Au(t,x),\ u(0,x)=f(x),\ u_t(0,x)=g(x)\,, \tag{1.1}$$

$$A=a^{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+b^i(x)\frac{\partial}{\partial x_i}+c(x),\ x=(x_1,\cdots,x_m)\,.$$

The problem is equivalent to the matricial equation

$$\frac{\partial}{\partial t}\begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix}=\begin{pmatrix}0 & I\\ A & 0\end{pmatrix}\begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix},\quad \begin{pmatrix}u(0,x)\\ v(0,x)\end{pmatrix}=\begin{pmatrix}f(x)\\ g(x)\end{pmatrix}, \tag{1.1$'$}$$

and we may apply the theory of semi-group of linear operators[1] to the integration in the large of (1.1), by considering, in a suitable Banach space, the "resolvent equation"

$$\left(\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}-n^{-1}\begin{pmatrix}0 & I\\ A & 0\end{pmatrix}\right)\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}f\\ g\end{pmatrix}\quad \text{for large } |n|,\quad (n=\text{integer}) \tag{1.2}$$

and obtaining the estimate

$$\left\|\begin{pmatrix}u\\ v\end{pmatrix}\right\|\leqq(1+|n^{-1}|\beta)\left\|\begin{pmatrix}f\\ g\end{pmatrix}\right\| \tag{1.3}$$

where β is a positive constant independent of n, f and g. The irrelevance of the sign of n implies that

$$\begin{pmatrix}0 & I\\ A & 0\end{pmatrix} \tag{1.4}$$

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1) E. Hille: Functional Analysis and Semi-groups, New York (1948).
K. Yosida: On the differentiability and the representation of one-parameter semi-group of linear operators, J. Math. Soc. Japan, 1 (1948), 15-21.