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A metamathematical theorem on functions.

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In our former paper [2], [3], we have introduced a logical system *GLC* and a subsystem G^1LC of *GLC*, as generalizations of Gentzen's *LK* (cf. [1]). We have also defined the notion of functions in *GLC* in [2]. This paper is most related to [3], where we have dealt with G^1LC without bound functions. We shall introduce in this paper another logical system called *HLC* ('hierarchical' logic calculus) lying between G^1LC and *LK* (§ 1). We shall define also 'functionals' in generalization of the notion of functions.

The purpose of the present paper is to prove that the consistent system under G^1LC without bound function or under *HLC* remains consistent after 'adjunction' of the concept of functionals, under certain conditions. Our Main Theorem will read as follows:

MAIN THEOREM: *Let Γ_0 be a system of axioms consistent under G^1LC without bound function or under HLC. Suppose Γ_0 contains axioms of equality* (See § 1 for definition), *and let the following sequences be provable.*

$$\Gamma_0 \to \forall\varphi_1\cdots\forall\varphi_n\forall x_1\cdots\forall x_m\exists yF(\varphi_1,\cdots,\varphi_n,x_1,\cdots,x_m,y)$$

$$\Gamma_0 \to \forall\varphi_1\cdots\forall\varphi_n\forall x_1\cdots\forall x_m\forall y\forall z(F(\varphi_1,\cdots,\varphi_n,x_1,\cdots,x_m,y)$$
$$\wedge F(\varphi_1,\cdots,\varphi_n,x_1,\cdots,x_m z)\vdash y=z)\,.$$

Let M be a functional not contained in Γ_0, and suppose further, in case of *HLC*, that $F(\alpha_1,\cdots,\alpha_n,a_1,\cdots,a_m,b)$ does not contain $\forall$ on f-variables. Then Γ_0 and the following axiom are consistent.

$$\forall\varphi_1\cdots\forall\varphi_n\forall x_1\cdots\forall x_m F(\varphi_1,\cdots,\varphi_n,x_1,\cdots,x_m,M(\varphi_1,\cdots,\varphi_n,x_1,\cdots,x_m))\,.$$

The conclusion of this theorem holds also in *LK* by theorem 2, proved in § 1.

After some preparations in § 1, we shall prove our main theorem