Journal of the Mathematical Society of Japan Vol. 7, No. 4, October, 1955

Homogeneous Riemannian spaces of four dimensions.

By Shigeru ISHIHARA

(Received Sept. 12, 1954)

E. Cartan [1] gave a general method to determine homogeneous Riemannian spaces, i. e. those which admit a transitive group of isometries or motions. He determined, in particular, applying his own method, the topological structure of three dimensional homogeneous Riemannian spaces.

On the other hand, K. Yano [10] has given many beautiful theorems on groups of isometries in a Riemannian space. But unfortunately four dimensional spaces are excluded in most of his theorems.

The purpose of the present paper is to determine, by the same method as Cartan's, the topological structure of four dimensional homogeneous Riemannian spaces which are connected and simply connected. Following the programme indicated in [1], we shall first determine in § 1 all the types of subgroups of the proper orthogonal group $R(4)$ in four variables. In § 2, we shall explain, for the sake of completeness, the general method of E. Cartan [1]. In § 3, we shall determine homogeneous Riemannian spaces admitting each subgroup obtained in § 1 as the group of stability. In most cases we determine the local structure of homogeneous Riemannian spaces by the method of moving frames, and from the local structure obtained we shall find the topological structure of the spaces.

Our main result is the following

THEOREM. *Any four dimensional homogeneous Riemannian space which is connected and simply connected is homeomorphic to one of the following manifolds:*

Euclidean space of four dimensions,
Sphere of four dimensions,
Complex projective space of two complex dimensions,
Product space of two spheres of two dimensions,