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Arithmetic of Orthogonal Groups.

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Three decades ago, H. Hasse discovered in his fundamental research on the arithmetic of quadratic forms[1] a remarkable principle, which may be stated as follows in its most general form: Let K be an abstract field, k an algebraic number field or a field of algebraic functions of one variable over a finite field, $\mathfrak{p}$ a place of k and $k_{\mathfrak{p}}$ the $\mathfrak{p}$-completion of k. Let furthermore $O(K)$ be an object related to K and Π a property of $O(K)$ (e.g. $O(K)$ may be a quadratic form f, and Π the representability of zero by f). If K is in particular specialized to k, then $O(K)$ will represent an object $O(k)$ related to k. Let $O_{\mathfrak{p}}$ be the object constructed naturally from $O(k)$ by transition to $k_{\mathfrak{p}}$ from k. By the 'Hasse principle' we shall mean this statement: *The assertion '$O(k)$ has the property Π' follows from the fact that '$O_{\mathfrak{p}}$ has the property Π for every place $\mathfrak{p}$ of k'.* Theorems of this form will be called theorems of Hasse type. Hasse has proved some important theorems of this type, concerning quadratic forms, cyclic extensions of algebraic number field, etc.[2]

Now the question arises: to what extent is this principle valid? In the present paper we shall investigate this question in connection with the orthogonal groups, as they are in closest relation to quadratic forms. In fact these groups can be treated in the same time with the corresponding quadratic forms by the geometrical method of Dieudonné, Eichler[3] and others.

Thus, in § 1 we introduce preliminary notions such as the similarity of forms, the indices and coindices of forms and the Clifford algebra of forms. We prove a lemma (Lemma 1) which shows that the similarity of forms is more natural than the congruence from geometric-

1) Hasse [1] and [2].
2) Deuring [3] Chap. VII and Witt [4].
3) Dieudonné [5] and Eichler [6].