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A simple proof of Dirichlet principle.

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1. In the usual proof of the Dirichlet principle, the solvabilty of the Dirichlet problem is not assumed. But since this can be proved simply by Perron's method, if we assume this, then we can prove the Dirichlet principle simply, which we shall show in the following lines. First we shall prove two lemmas.

LEMMA 1. *Let $w(z)$ be continuous in a ring domain $\Delta: 0<\rho\leqq|z|\leqq 1$ and have piece-wise continuous partial derivatives of the first order and its Dirichlet integral $D_{\Delta}[w]=D[w]$ be finite. Let $u(z)$ be harmonic in $\rho<|z|<1$ and continuous in $\rho\leqq|z|\leqq 1$, such that $u(z)=w(z)$ on $|z|=\rho$ and $|z|=1$. Then*

$$D[u]\leqq D[w].$$

PROOF. Let in $\rho<|z|<1$,

$$u(z)=u(re^{i\theta})=A\log r+a_0+\sum_{k=1}^{\infty}(a_k r^k+a_{-k}r^{-k})\cos k\theta + \sum_{k=1}^{\infty}(b_k r^k+b_{-k}r^{-k})\sin k\theta\,, \tag{1}$$

where

$$\left.\begin{aligned} a_0&=\frac{1}{2\pi}\int_0^{2\pi}u(e^{i\theta})d\theta=\frac{1}{2\pi}\int_0^{2\pi}w(e^{i\theta})d\theta\,,\\ a_k+a_{-k}&=\frac{1}{\pi}\int_0^{2\pi}u(e^{i\theta})\cos k\theta\, d\theta=\frac{1}{\pi}\int_0^{2\pi}w(e^{i\theta})\cos k\theta\, d\theta\,,\\ b_k+b_{-k}&=\frac{1}{\pi}\int_0^{2\pi}u(e^{i\theta})\sin k\theta\, d\theta=\frac{1}{\pi}\int_0^{2\pi}w(e^{i\theta})\sin k\theta\, d\theta\,,\end{aligned}\right\} \tag{2}$$