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On the fundamental theorem of algebra.

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In this note we give an elementary proof for the fundamental theorem of algebra that the complex number field C is algebraically closed, using a normed-ring-theoretic method.

For this purpose let $C[x]$ be the polynomial ring over C. We define the absolute value of $f\in C[x]$ by $|f|=|a_0|+\cdots+|a_m|$ where $f=a_0+\cdots+a_m x^m$, so that $|f|\geqq 0$ always, and $|f|=0$ if and only if $f=0$. (This symbol is clearly compatible with the usual absolute value when $f\in C$.) It follows easily that for $f, g\in C[x]$ and $z\in C$

$$|f+g|\leqq|f|+|g|, \qquad |fg|\leqq|f|\cdot|g|, \qquad |zf|=|z|\cdot|f|.$$

Suppose that $\phi\in C[x]$ is a fixed monic polynomial of degree $n\geqq 1$. We define an operator Φ for $f\in C[x]$ as follows. If $f=0$ or $\deg f<n$, we put $\Phi f=f$; if $m=\deg f\geqq n$ and $f=a_0+\cdots+a_m x^m$, then we put $\Phi f=f-a_m x^{m-n}\phi$, so that $\deg \Phi f<\deg f$ for $\Phi f\neq 0$ in this latter case. Then we clearly have always

$$|\Phi f|\leqq|f|+|f|\cdot|\phi|=M|f| \qquad (M=|\phi|+1),$$

and so $|\Phi^n f|\leqq M^n|f|$.

Now let ϕ considered above be irreducible over C. To prove the theorem, it then suffices to show that $n=1$. The residue-classes of $C[x]$ modulo ϕ form a field E, which contains C as a subfield if we identify each $z\in C$ with the residue-class containing z. Let $\theta\in E$ be the residue-class represented by x, then $\phi(\theta)=0$ and for each $\alpha\in E$ there is a uniquely determined polynomial $f_\alpha\in C[x]$ such that $\alpha=f_\alpha(\theta)$ and that the degree of f_α is $<n$ when $\alpha\neq 0$.

This being so, we define $|\alpha|=|f_\alpha|$ for $\alpha\in E$, so that $|\alpha|\geqq 0$ always, $|\alpha|=0$ if and only if $\alpha=0$, and $|z\alpha|=|z|\cdot|\alpha|$ for $z\in C$. (This symbol coincides with the usual absolute value when $\alpha\in C$.) If $\alpha, \beta\in E$, it is easily seen that $f_{\alpha+\beta}=f_\alpha+f_\beta$, whence $|\alpha+\beta|\leqq|\alpha|+|\beta|$. Further,