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Star-like theorems and convex-like theorems in an annulus.

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1. Introduction.

It is the purpose of this paper to obtain some sufficient conditions for p-valency of $f(z)$, meromorphic and single-valued in an annulus.

It is well known that if $f(z)=z+a_2z^2+\cdots$, regular for $|z|<r$, satisfies the relation $\Re[zf'(z)/f(z)]>0$ for $|z|<r$, then $f(z)$ is univalent in $|z|<r$ [5]. This theorem has been generalized considerably by S. Ozaki [1,2] and the author [3]. We shall give in § 2 generalizations of the above theorems to the case of meromorphic functions defined in an annulus.

In § 3, we shall study analytic functions convex in one direction in a ring domain, which will be defined precisely later on, extending the results given by the author [4].

2. Star-like theorems.

THEOREM 1. *Let $f(z)$ be meromorphic and single-valued in $r\leqq|z|\leqq R$ and let $n(0)$ and $n(\infty)$ be the number of zeros and poles of $f(z)$ in $r\leqq|z|\leqq R$, respectively. Further let*

$$\frac{1}{2\pi}\int_{|z|=R}\Re\frac{zf'(z)}{f(z)}\,d\theta=p\,,\qquad \frac{1}{2\pi}\int_{|z|=r}\Re\frac{zf'(z)}{f(z)}\,d\theta=q\,.$$

(i) *If $\Re[zf'(z)/f(z)]>0$ on $|z|=R$ and $|z|=r$, then $f(z)$ is at most $p+n(\infty)$ $\big(=q+n(0)\big)$-valent and at least* Max $[n(0)-p,1]$*-valent in $r\leqq|z|\leqq R$.*

(ii) *If $\Re[zf'(z)/f(z)]<0$ on $|z|=R$ and $|z|=r$, then $f(z)$ is at most $n(0)-p\big(=n(\infty)-q\big)$-valent and at least* Max $[n(\infty)+p,1]$*-valent in $r\leqq|z|\leqq R$.*