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Symmetrization and univalent functions in an annulus.

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1. Introduction and notations.

Recently the theory of symmetrization due to Pólya and Szegö has successfully been utilized in the theory of functions and potential theory by Hayman [3] and Jenkins [4]. In the present paper we will, by the method of symmetrization, obtain several results on omitted values of univalent functions in an annulus, which may be considered as extensions of theorems established by Goodman [1] and Jenkins [4].

For the purpose we take an annulus in the z-plane

$$D: \quad Q<|z|<1 \qquad (Q>0)$$

as a doubly-connected basic domain and consider a class $\mathfrak{F}$ of univalent functions $w=f(z)$ which are regular in D and map D onto subdomains of the domain $|w|>Q$ in such a way that the circle $|w|=Q$ corresponds to the circle $|z|=Q$.

In the sequel the Grötzsch's extremal function [2]

$$(1) \qquad w_q=f_0(z, q), \qquad f_0(Q, q)=q \qquad (0<q<Q),$$

which maps D onto an annulus $q<|w_q|<1$ slit from $w_q=1$ to $w_q=\omega_q$ $(q<\omega_q<1)$ along the positive real axis, plays an important rôle and it is explicitly represented in terms of the elliptic function $\wp(u)$ in the form [6]

$$(2) \quad k'(q)^2\frac{\wp_q\left(\frac{1}{i}\lg w_q\right)-e_3(q)}{\wp_q\left(\frac{1}{i}\lg w_q\right)-e_2(q)}=k'(Q)^2\frac{\wp_Q\left(\frac{1}{i}\lg z\right)-e_3(Q)}{\wp_Q\left(\frac{1}{i}\lg z\right)-e_2(Q)},$$

the primitive periods of $\wp_q(u)$ being 2π and $2i\lg(1/q)$, and $k'(q)$ being a complementary modulus of the elliptic function sn. For the brevity