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A generalization of Riesz-Fischer's theorem.

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L. Kantorovitch [1] has proved, generalizing Riesz-Fischer's theorem to an abstract form, that a normed semi-ordered linear space is complete if the norm satisfies the following two conditions:

(1) The norm is monotone complete, i.e., every non-decreasing and norm-bounded sequence of positive elements has the least upper bound.

(2) The norm is continuous, i.e., every non-increasing sequence of elements which is order-convergent to 0 also converges to 0 by the norm.

H. Nakano [2] has shown that the condition (2) can be weakened to the condition of semi-continuity, i.e.,

(2′) For every non-decreasing sequence of positive elements, $a_\nu \uparrow_{\nu=1}^{\infty}$, such that $a=\bigcup_{\nu=1}^{\infty} a_\nu$ exists, we have

$$\|a\|=\sup_{\nu\geq 1}\|a_\nu\|.$$

Here we shall show that even this condition (2′) is superfluous, i.e. only the condition (1) suffices for the completeness of the normed semi-ordered linear space.

First we shall prove the following

LEMMA. *If a normed semi-ordered linear space R has a monotone complete norm, then there exists a positive real number $\alpha\leqq 1$ such that $0\leqq a_\nu \uparrow_{\nu=1}^{\infty} a$ implies always*

$$\sup_{\nu\geq 1}\|a_\nu\|\geqq\alpha\|a\|.$$

PROOF. If we can not find such a number α, then there exists a double sequence of positive elements $a_{\mu,\nu}$ and a sequence a_μ ($\mu=1,2,\cdots$), such that we have

$$0\leqq a_{\mu,\nu}\uparrow_{\nu=1}^{\infty} a_\mu, \qquad \|a_\mu\|\geqq\mu \quad \text{and} \quad \sup_{\nu\geq 1}\|a_{\mu,\nu}\|\leqq\frac{1}{2^\mu}.$$