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Concave modulars.

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We have defined and discussed modulars on semi-ordered linear space in a book[1]. Let R be a semi-ordered linear space and universally continuous, that is, for every system of positive elements $a_\lambda \in R$ $(\lambda \in \Lambda)$ there exists $\bigcap_{\lambda \in \Lambda} a_\lambda$. A functional $m(x)$ $(x \in R)$ is called a *modular* on R, if 1) $0 \leqq m(x) \leqq +\infty$ for every $x \in R$, 2) $m(\xi a)=0$ for every $\xi \geqq 0$ implies $a=0$, 3) for any $a \in R$ we can find $\alpha > 0$ such that $m(\alpha a) < +\infty$, 4) for each $x \in R$, $m(\xi x)$ is a convex function of ξ: $m\left(\frac{\alpha+\beta}{2} x\right) \leqq \frac{1}{2}\{m(\alpha x)+m(\beta x)\}$, 5) $|x| \leqq |y|$ implies $m(x) \leqq m(y)$, 6) $x \frown y=0$ implies $m(x+y)=m(x)+m(y)$, 7) $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x_0$ implies $m(x_0)=\sup_{\lambda \in \Lambda} m(x_\lambda)$.

In this paper we shall consider a functional $m(x)$ $(x \in R)$ which satisfies instead of 4) the condition: $m(\xi x)$ is a concave function of $\xi \geqq 0$, i.e., we define a *concave modular* $m(x)$ $(x \in R)$ by the postulates: 1) $0 \leqq m(x) < +\infty$, 2) $m(x)=0$ implies $x=0$, 3) $|x| \leqq |y|$ implies $m(x) \leqq m(y)$, 4) $x \frown y=0$ implies $m(x+y)=m(x)+m(y)$, 5) $m(\xi x)$ is a concave function of $\xi \geqq 0$:

$$m\left(\frac{\lambda+\mu}{2} x\right) \geqq \frac{1}{2}\{m(\lambda x)+m(\mu x)\} \qquad \text{for } \lambda, \mu \geqq 0,$$

6) $\lim_{\xi \to 0} m(\xi x)=0$, 7) $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$, $\sup_{\nu \geqq 1} m(x_\nu) < +\infty$ implies the existence of an element x_0 for which $x_\nu \uparrow_{\nu=1}^{\infty} x_0$ and $m(x_0)=\lim_{\nu \to \infty} m(x_\nu)$.

Concerning the concave modulars $m(x)$ on R, we can prove

$$m(x+y) \leqq m(x)+m(y) \qquad \text{for every } x, y \in R.$$

Thus, every concave modular $m(x)$ on R is a quasi-norm by which R is a Fréchet space.

For a concave modular $m(x)$ on R, we can prove easily