Journal of the Mathematical Society of Japan Vol. 4, Nos. 3~4, December, 1952.

On the multivalency of analytic functions.

By Toshio UMEZAWA

(Received January 10, 1952)

(Revised February 18, 1952)

Noshiro's theorems[1)] (generalizations of Dieudonné's theorem[2)]) concerning the univalency of regular functions were extended to the case of p-valence by E. Sakai[3)]. In the present paper we are going to generalize some of them to meromorphic functions which are defined in a multiply-connected domain. By accomplishing this task we shall also be able to extend Z. Nehari's results[4)] and to make them more sharp.

LEMMA 1. *Let $\varphi(z)$ be regular in an n-ply connected domain D and let $\varphi(z) \subset T$*[5)] *in D, where T is a given connected domain. Let us denote by $u=g(t)$ an arbitrary branch of a function mapping T conformally on $|u|<1$ and suppose that $g(\varphi(z))$ is single-valued in D, and put*

$$(1) \qquad \frac{1-|g(\alpha)|^2}{|g'(\alpha)|} \equiv \Omega(\alpha, T) \qquad (\alpha \in T)^{6)}.$$

Then

$$(2) \qquad |\varphi'(z)| \leqq 2\pi k(z, z)\Omega(\varphi(z), T) \qquad (z \in D),$$

where $k(z, \xi)$ denotes the Szegö kernel function[7)] *of D.*

PROOF. In order that the integration be permissible we assume that the boundary Γ of D consists of smooth curves and that $\varphi(\zeta)$ is continuous on Γ; but once the result is obtained, both assumptions can easily be disposed of. Indeed, if D is not smoothly bounded, we may approximate D by a sequence of domains D_n which satisfy $D_n \subset D$, $D_n \subset D_{n+1}$, $\lim_{n\to\infty} D_n = D$ and whose boundaries Γ_n are smooth. If we replace D by D_n, the additional assumption under which we prove Lemma 1 are satisfied. The general result then follows by letting $n \to \infty$ and observing that the Szegö kernel function $k(z, z)$ is a continuous domain function.