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A metamathematical theorem on the theory of ordinal numbers.

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The purpose of this paper is to show that the consistency proof of a theory of ordinal numbers in the weakened form considered in G. Gentzen's logical system LK (cf. Gentzen [1]) can be reduced to that of a weakened theory of ordinal numbers $< \omega^{\omega}$, this latter theory being considered in a logical system which is obtained in extending slightly the system LK by the use of the symbol Min: if $\mathfrak{A}(a)$ is a formula and x is any bound variable not contained in $\mathfrak{A}(a)$, the figure Min$(x)\mathfrak{A}(x)$ is a 'term', a figure for a particular object. (We follow the terminology of Gentzen [1].) What these theories mean, will be described below by sets of axioms 1.1,$\cdots$, 1.16 and 2.1,$\cdots$, 2.19 respectively. Thus, we shall prove that any set of axioms indicated in 1.1,$\cdots$, 1.16, containing no special object other than 0, ω and no function other than $*'$ ($*$ indicates an argument-place), is consistent, in assuming that any set of axioms indicated in 2.1,$\cdots$,2.19 can not lead to a contradiction.

To perform this, we shall establish a metatheorem called Representation Theorem, which is meaningful in the weakened theory of ordinal numbers.

Each of 1.12,$\cdots$, 1.16, 2.16,$\cdots$, 2.19 stands for a finite number of arbitrary axioms of the indicated form, and $[z]$ stands for a row of symbols of the form $\forall z_1 \cdots \forall z_k$; properly we should write $\mathfrak{A}(x, z_1, \cdots, z_k)$ or $\mathfrak{A}(x, y, z_1, \cdots, z_k)$ for $\mathfrak{A}(x)$ or $\mathfrak{A}(x, y)$ respectively, but it seems improbable that any confusion should occur from our simplified expression.

Some metamathematical lemmas, e. g. the one formulated immediately below, will be useful in our consideration, but it seems unimportant to give all such lemmas used, which are merely explicit and rather long formulation of mathematicians' common sense.

LEMMA. Let $\mathfrak{M}_i (i=0, 1)$ be two formulas obtained exactly in the