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An elementary proof of the fundamental theorem of normed fields.

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In the theory of normed rings founded by I. Gelfand[1], the most fundamental is the following theorem of Mazur-Galfand: "A normed field over the complex number field is, in the sense of isomorphism, nothing but the complex number field itself", the proof of which was depending on the notion of analyticity, Cauchy's integral formula and Liouville's theorem, etc[2].

Quite recently, Prof. E. Artin in his lecture note 'Algebraic numbers and algebraic functions I (1950–51) Princeton' gives a proof of this theorem as one of the bases of his theory, replacing contour integral by its approximating sum.

The aim of this short note is also to give an elementary proof of the above theorem, using no function-theoretical methods but the notion of continuity. Moreover the proof does not assume the completeness of the normed fields, though it is easy to see, by way of completion, that the assumption of completeness does not harm the generality of this theorem.

Let K be a normed field, that is to say, a field in the sence of algebra and at the same time a linear space, over the complex number field C, in which is given the norm $\| \ \|$ satisfying

$$x \in K \rightarrow \|x\| \geqq 0,$$

$$x \neq 0 \rightleftarrows \|x\| > 0,$$

$$x, y \in K \rightarrow \|x+y\| \leqq \|x\| + \|y\|, \quad \|xy\| \leqq \|x\| \cdot \|y\|,$$

$$\lambda \in C,\ x \in K \rightarrow \|\lambda x\| = |\lambda| \cdot \|x\|.$$

Then the unit e of K, being $\neq 0$, has positive norm, and consequently, $\|e\| \geqq 1$.

1) I. Gelfand: "Normierte Ringe", Recueil Math. T. 9 (51) No. 1 (1941).
2) ibid.