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On the zeros of integral functions of integral order.

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1. Let $f(z)$ be an integral function of integral order $\rho > 0$, and $M(r)$ be its maximum modulus on the circumference $|z|=r$. Further, let $n(r, \alpha)$ denote the number of zeros of $f(z)-\alpha$ for any complex α. In this note we shall prove the following two theorems:

THEOREM 1. *If* $\log_2 M(r)/\log r$ *has the limit* ρ *for* $r \to \infty$, *then* $\log n(r, \alpha)/\log r$ *has the same limit for* $r \to \infty$, *except possibly for some values of* α *belonging to a set of inner logarithmic capacity zero.*

THEOREM 2. *If* $\log M(r)/r^{\rho}$ *is bounded from zero and infinity, so is* $n(r, \alpha)/r^{\rho}$, *except possibly for some values of* α *belonging to a set of inner capacity zero.*

It is known that these theorems hold with an exceptional set whose projection on any straight-line is of zero content[1)].

Our proof is based on the following well-known fact:

LEMMA 1. *For an integral function of non-integral order, above two theorems hold for any* α *without exception*[1)].

2. Let $f(z)$ be meromorphic in $|z| < +\infty$ and s be a positive integer. We put

$$F_{\alpha}(W)=\prod_{k=0}^{s-1}\left[f(zt^{k})-\alpha\right], \quad W=z^{s} \text{ and } R=r^{s},$$

where t is a primitive s-th root of 1, so that $F_{\alpha}(W)$ is meromorphic in $|W| < +\infty$. Then,

LEMMA 2. *There holds*

$$T(R, F_{\alpha}) \sim sT(r, f)$$

except possibly for some values of α *belonging to a set of inner capacity zero.*