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On maximum modulus of integral functions.

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Let D be a region on the z-plane, which lies in the disc $|z|<R$ $(0<R\leqq+\infty)$, and whose boundary Γ lying in $|z|<R$ consists of a finite or infinite number of analytic curves clustering nowhere in $|z|<R$. For any $0<r<R$, we denote by D_r the part of D lying in $|z|<r$. Let $A_k(r)$ $\big(k=1, \cdots, n(r)\big)$ be the arcs of $|z|=r<R$ contained in D, and $r\cdot\theta_k(r)$ be their lengths.

We define a function $\theta(r)$ in $0<r<R$ as follows: if $|z|=r$ is contained wholly in D, then $\theta(r)=+\infty$, and, otherwise, $\theta(r)=\max\limits_k \theta_k(r)$.

Using Carleman's method[1], we shall first prove

THEOREM 1. *Suppose that* $\theta(r)>0$ *for* $0<r_0<r<R$, *and let* $u(z)$ *be a harmonic function in* D, *which is* >0 *in* D *and* $=0$ *on* Γ. *We put*

$$m(r)=\frac{1}{2\pi}\sum_k\int_{A_k(r)}\Big[u(re^{i\varphi})\Big]^2 d\varphi \qquad (0<r<R)$$

and $$D(r)=\iint_{D_r}\left[\left(\frac{\partial u}{\partial \log r}\right)^2+\left(\frac{\partial u}{\partial \varphi}\right)^2\right]d\log r\, d\varphi\,.$$

Then, for any $0<r_0<r<R$,

$$D(r)\geqq D(r_0)\exp.\int_{r_0}^{r}\frac{2\pi}{r\theta(r)}dr$$

and $$m(r)-m(r_0)\geqq\frac{1}{\pi}D(r_0)\cdot\int_{r_0}^{r}\frac{dt}{t}\left[\exp.\int_{r_0}^{t}\frac{2\pi}{s\theta(s)}ds\right].$$

Let $f(z)$ be a regular analytic function in $|z|<R\leqq+\infty$. While applying Theorem 1 to $u(z)=\overset{+}{\log}|f(z)|$, we shall obtain some theorems on the modulus of $f(z)$.

PROOF OF THEOREM 1. Since $u=0$ on Γ, we have, by application of Green's formula,