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A proof of a transformation formula in the theory of partitions.

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Let us denote by $p(n)$ the number of unrestricted partitions of a positive integer n. Then we have

$$f(x)=1+\sum_{n=1}^{\infty} p(n)x^n=\prod_{n=1}^{\infty}\frac{1}{1-x^n},$$

where $f(x)$ is defined and regular for $|x|<1$. Now this function $f(x)$ is known to satisfy a remarkable transformation formula (see Rademacher [3]):

$$f\left(e^{\frac{2\pi ih}{k}-\frac{2\pi z}{k}}\right)=\omega_{h,k}\sqrt{z}\,e^{\frac{\pi}{12kz}-\frac{\pi z}{12k}}f\left(e^{\frac{2\pi iH}{k}-\frac{2\pi}{kz}}\right), \tag{1}$$

where h, k and H are positive integers such that

$$(h,k)=1, \qquad hH\equiv -1 \pmod{k},$$

and

$$\omega_{h,k}=\exp\left(\pi i\sum_{m=1}^{k-1}\frac{m}{k}\left(\frac{hm}{k}-\left[\frac{hm}{k}\right]-\frac{1}{2}\right)\right),$$

an empty sum meaning zero; further, z is a complex variable with positive real part and we take the principal branch as the determination of $\sqrt{z}$.

The formula (1) was used by Hardy-Ramanujan, and also by Rademacher subsequently, in their famous researches [2] and [4] on the function $p(n)$.

It is the main object of this paper to give a proof of (1) which is directly based on the following well known transformation formula in the theory of elliptic theta-functions:

$$\sum_{n=-\infty}^{\infty} e^{-(n+\alpha)^2\pi t}=\frac{1}{\sqrt{t}}\sum_{n=-\infty}^{\infty} e^{-\frac{n^2\pi}{t}}\cos(2\pi n\alpha), \tag{2}$$