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A Deformation Theorem on Conformal Mapping.

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We will prove the following deformation theorem on conformal mapping.

Theorem 1. *Let D be a simply connected domain on the z-plane, which contains $z=0$ and is contained in $|z|<M$. Let E be a continuum, which contains $z=0$ and is contained in D, such that a disc of radius ρ about any point of E is contained in D. If we map D conformally on $|w|<1$ by $w=w(z)$, $z=z(w)$ $(w(o)=o))$, then the image of E is contained in $|w|<1-k<1$, where $k=k\left(\frac{\rho}{M}\right)$ depends on $\frac{\rho}{M}$ only.*

We can take

$$k=\frac{\rho}{4M}e^{-a\frac{M^2}{\rho^2}}\left(a=\frac{64\pi}{\sqrt{3}}\log\frac{32}{9}<100\right).$$

Proof. We cover the z-plane by a net of regular triangles Δ_i of sides $\frac{\rho}{4}$, whose vertices are $z_{m,n}=m\frac{\rho}{4}e^{\frac{\pi i}{3}}+n\frac{\rho}{4}(m,\ n=0,\ \pm1,\ \pm2,\cdots\cdots)$. It is easily seen that if Δ_i contains a point of E, then a disc of radius $\frac{3\rho}{4}$ about a vertex ζ_i of Δ_i is contained in D, so that Δ_i is contained in D and $w(z)$ is regular and schlicht in $|z-\zeta_i|<\frac{3\rho}{4}$. Let $\Delta_1, \Delta_2,\cdots\cdots,\Delta_N$ be the triangles which contain points of E, where $z=0$ is a vertex of Δ_1, then since the area of Δ_i is $\frac{\sqrt{3}\rho^2}{64}$ and is contained in $|z|<M$,

$$N<\mu=\frac{64\pi}{\sqrt{3}}\frac{M^2}{\rho^2}. \tag{1}$$

Let z_0 be any point of E and let z_0 be contained in Δ_{n_0} $(n_0\leqq N)$, then since E is a continuum, there exists a chain of triangles:

$$\Delta_1,\ \Delta_2,\cdots\cdots,\ \Delta_{n_0} \qquad (n_0\leqq N),$$

where Δ_i, Δ_{i+1} have a common side, so that $|\zeta_i-\zeta_{i+1}|=\frac{\rho}{4}$ and each Δ_i con-