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On the differentiability and the representation of one-parameter semi-group of linear operators.

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1. *The theorems.* Let $\{U_t\}$, $0 \leqq t < \infty$, be a one-parameter semi-group of linear (=additive, continuous) operators from a complex Banach space E to E:

$$U_t U_s = U_{t+s}, \quad U_0 = I \quad (=\text{the identity operator}), \tag{1·1}$$

such that

$$\sup_t \| U_t \| \leqq 1, \tag{1·2}$$

$$\lim_{t \to t_0} U_t x = U_{t_0} x \quad (\lim = \text{strong limit}), \quad 0 \leqq t_0 < \infty, \ x \in E. \tag{1·3}$$

The purpose of the present note is to prove the following two theorems[1].

Theorem 1. If we denote by D the totality of x for which

$$\underset{h \downarrow 0}{\text{weak lim}}\ h^{-1}(U_h - I)x = Ax \tag{1·4}$$

exists, then D is dense in E. Moreover A is a closed additive operator from D to E with the properties:

$$\text{for any } x \in D, \ \lim_{h \to 0} h^{-1}(U_{t+h} - U_t)x = AU_t x = U_t Ax, \tag{1·5}$$

there exists a sequence $\{I_n\}$ of linear operators each commutative with every U_t and A such that i) the range $R(I_n) = \{I_n x\,;\ x \in E\} \subseteq D$, $AI_n = n(I_n - I)$, ii) $\| I_n \| \leqq 1$, $\lim_{n \to \infty} I_n x = x$, iii) $U_t x = \lim_{n \to \infty} \exp(tAI_n)x = \lim_{n \to \infty} \sum_{m=0}^{\infty} (m!)^{-1}(tAI_n)^m x$ uniformly for t in any finite interval[2], (1·6)

$\| (A - nI)x \| \geqq n \| x \|$ $(n = 1, 2, \ldots\ldots)$ *for $x \in D$ and the range $R(A - nI)$ coinsides with E $(n = 1, 2, \ldots\ldots)$,* (1·7)

let, by (1·7), y_n be the unique solution of $(A - nI)y_n = y$ $(n = 1, 2, \ldots\ldots)$, then $\lim_{n \to \infty} A(-ny_n) = \lim_{n \to \infty} (-n(y + ny_n)) = Ay$ for $y \in D$. (1·8)

Theorem 2. Let, conversely, A be an additive operator from a dense