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The first derived limit and compactly F_σ sets

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For a sequence $a = \{a(n)\}$ of integers (a member of $\boldsymbol{N}^{\boldsymbol{N}}$), set

$$A_a = \bigoplus_{m=1}^{\infty} \bigoplus_{n=1}^{a(m)} \boldsymbol{Z}.$$

One can view A_a as the group of all functions from the set $D_a = \{(m,n) : n \leq a(m)\}$ in $\boldsymbol{Z}$ which are equal to 0 on all but finitely many pairs (m,n) in D_a. Considering $\boldsymbol{N}^{\boldsymbol{N}}$ as a directed set ordered by the product-ordering ($a \leq b$ iff $a(n) \leq b(n)$ for all n), we get an inverse system

$$\mathscr{A} = \langle A_a, \pi_a^b, \boldsymbol{N}^{\boldsymbol{N}} \rangle$$

of Abelian groups, where $\pi_a^b : A_b \longrightarrow A_a$ are the natural projections. The first derived limit

$$\underleftarrow{\lim}^{(1)} \mathscr{A}$$

of this inverse system is an object of considerable interest in several areas of mathematics ([**14**], [**4**], [**9**], [**10**], [**6**; §8]). The purpose of this short note is to connect it with yet another area, descriptive set theory. The problem we consider was originally asked by Jayne and Rogers and formulated in its present form by Fremlin ([**11**], [**1**; 230 (d)], [**2**; DI]). The original question of J. E. Jayne and C. A. Rogers states whether for a given Polish space M and analytic subset X of M which is not Borel there is always a compact subset K of M such that $X \cap K$ is not Borel. This of course leads to similar questions about other classes of sets of reals and the way they behave from 'the point of view of compact sets' (see [**11**], [**12**], [**18**]). For example, the role of Martin's axiom and the negation of the continuum hypothesis (or more precisely the role of the boundedness number of the ordering of eventual dominance in $\boldsymbol{N}^{\boldsymbol{N}}$) in finding positive answers to these kind of questions has been recognized very early ([**11**]). This was the motivation behind the problem (which we solve here) whether similar assumptions are also sufficient to answer the Jayne–Rogers question without the restriction that the set X is analytic ([**1**; 230 (d)], [**2**; DI]). In our proofs we shall use ideas from a few different subjects. For background on the homological algebra needed for this paper the reader is referred to [**4**]. The background on descriptive set theory can be found in [**17**], while the background on forcing axioms can be found in [**1**], [**5**], [**6**], [**7**], and [**15**]. The basic notions and facts from topology can be found in [**19**].

§1. A compactly-simple set.

THEOREM 1. *If* $\underleftarrow{\lim}^{(1)} \mathscr{A} \neq 0$ *then there is a subset* X *of* $\boldsymbol{R} \backslash \boldsymbol{Q}$ *which is not analytic but its intersection with every compact subset of* $\boldsymbol{R} \backslash \boldsymbol{Q}$ *is* F_σ.