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The theorem of E. Hopf under uniform magnetic fields

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Introduction.

Let (M, g) be a complete Riemannian manifold, and let B be a closed 2-form on M. B may be regarded as a magnetic field on M. Let $\Omega : TM \to TM$ be a skewsymmetric matrix defined by $g_p(u, \Omega(v)) = B_p(u, v)$ $(u, v \in T_pM, p \in M)$. In [**8**], the Newtonian equation of a charged particle moving on M has been defined by

$$\frac{D}{dt}\dot{c} = \Omega(\dot{c}), \tag{1}$$

where $\dot{c}$ is the velocity vector field of the curve c and D/dt stands for the covariant derivative along c. *The magnetic flow* $\varphi_t : TM \to TM$ *associated with* B is defined by

$$\varphi_t(v) = \dot{c}_v(t),$$

where c_v is a solution curve of the equation (1) with $\dot{c}_v(0) = v \in TM$.

By using a geodesic flow, E. Hopf proved that *the total curvature of a compact surface without conjugate points is nonpositive, and vanishes if and only if the surface is flat* in [**5**]. L. W. Green extended the result of E. Hopf for a compact n-dimensional manifold in [**3**]. Recently, F. Guimarães ([**4**]) and N. Innami ([**6**]) have treated the noncompact case.

The non-existence of a pair of conjugate points along geodesics is equivalent to the non-existence of singular values of the exponential map. If there exists a magnetic field, then this equivalence no longer holds. From this fact, we find two concepts of non-conjugation for the magnetic flow, which are called *Jacobi field non-conjugation* and *exponential map non-conjugation*.

We call a magnetic field uniform if $\nabla B \equiv 0$, where ∇ is the Levi-Civita connection of M. In this paper, for each concept of non-conjugation, we will generalize E. Hopf's theorem with the help of the magnetic flow associated with a uniform magnetic field. In Section 3, the generalization for Jacobi field non-conjugation is treated. In Section 4, the following result will be proved as the generalization for exponential map non-conjugation.

THEOREM 1. *Let (M, g) be a compact orientable surface with a uniform magnetic field $B = b\,\mathrm{vol}_M$ $(b \in \boldsymbol{R})$ where vol_M is a canonical volume form of M, and let $\chi(M)$ denoted the Euler characteristic of M. Let $\exp^{\pm\Omega} : TM \to M$ be the exponential maps associated with B. Suppose that there exist no singular values of $\exp^{\pm\Omega}$. Then,*

$$-\frac{b^2}{2\pi}\,\mathrm{vol}(M) \geq \chi(M),$$

and the equality holds if and only if the curvature of M is constant $-b^2$.