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Ahlfors functions on non-planar Riemann surfaces whose double are hyperelliptic

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1. Introduction.

Let R be a finite bordered Riemann surface which is a regular subregion of a Riemann surface. The genus of R is finite and the boundary ∂R of R consists of a finite number of contours. Let $\mathfrak{F}$ be the class of holomorphic functions h on R satisfying $|h| < 1$ on R. Let P be a point on R. If $f \in \mathfrak{F}$ satisfies

$$|(f \circ \varphi^{-1})'(\varphi(P))| = \sup\{|(h \circ \varphi^{-1})'(\varphi(P))| : h \in \mathfrak{F}\}$$

for a fixed local parameter φ, then we call f the Ahlfors function at P on R. It is known, by Ahlfors [1], that the Ahlfors function exists at each point of R and is uniquely determined up to a constant multiple of absolute value 1. We note that the Ahlfors function does not depend on a choice of a local parameter φ.

We summarize some basic properties of Ahlfors functions. We define the subclass $\mathfrak{F}_1$ of $\mathfrak{F}$ as $\mathfrak{F}_1 = \{h \in \mathfrak{F} : |h| = 1 \text{ on } \partial R\}$. Then each Ahlfors function is an element of $\mathfrak{F}_1$, and gives a complete covering on the unit disk, that is, it covers each point of the unit disk the same number of times, provided that the branch points are counted as many times as their multiplicity indicates. The number will be called the degree of the Ahlfors function. Each Ahlfors function f is prime in $\mathfrak{F}_1$, that is, if $f = ab (a, b \in \mathfrak{F}_1)$, then either a or b is a constant of absolute value 1. The Ahlfors function at P vanishes to order 1 at P.

Let p be the genus of R and let q be the number of contours of R. Let N be the degree of an Ahlfors function on R. Then Ahlfors [1] showed that $q \leq N \leq 2p + q$. If $p = 0$, that is, if R is a planar region, then $N = q$. Let $N(R)$ be the set of degree of Ahlfors functions on R. Then $N(R) = \{q\}$, if $p = 0$. In this case, Ahlfors functions can be expressed by theta functions (see Fay [3] Proposition 6.17). On the other hand, if $p > 0$, then it is not well-known what is $N(R)$. There is only one example constructed by Yamada [4] (section 4. Example). Let Y be the example of a finite Riemann surface: the genus of Y is one and its boundary consists of two components. Yamada [4] (section 4. Example.) showed $\{2, 4\} \subset N(Y)$ in the example.

In this paper, we deal with the Ahlfors functions on non-planar Riemann surfaces whose double are hyperelliptic. In Section 2, we shall show that the double $\hat{R}$ of such Riemann surface R can be expressed as

$$y^2 = \prod_{j=1}^{g+1} (x - \alpha_j)(1 - \overline{\alpha_j} x), \tag{1}$$