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Ginzburg-Landau equation with magnetic effect: non-simply-connected domains

Dedicated to Professor Kôji KUBOTA on the occasion of his 60th birthday

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§ 1. Introduction and preliminaries.

We deal with the Ginzburg-Landau (GL) functional with its variational equation (GL equation) and study the existence of many kinds of local minimizers (stable solutions) in non-trivially geometrical situations. We consider the following GL functional:

$$\mathscr{H}_\lambda(\Phi, A) = \int_\Omega \left(\frac{1}{2}|(\nabla - iA)\Phi|^2 + \frac{\lambda}{4}(1 - |\Phi|^2)^2\right) dx + \int_{\boldsymbol{R}^3} \frac{1}{2}|\operatorname{rot} A|^2\, dx. \tag{1.1}$$

for the variable (Φ, A), where Φ is a $\boldsymbol{C}$-valued function in Ω and A is an $\boldsymbol{R}^3$-valued function in $\boldsymbol{R}^3$ and $\lambda > 0$ is a parameter. This type of functional appears in the theory of the (low-temperature) superconductivity (cf. [**18**]). Note that the first and second terms correspond to the energy of the electrons in the material Ω and that of the magnetic field, respectively. It should be emphasized that the magnetic field occurs in the whole space $\boldsymbol{R}^3$. The theory suggests that a physically realizable state corresponds to a local minimizer (of such an energy functional) and hence, in our case, it becomes a solution (Φ, A) to the following variational equation (1.2) (GL equation):

$$\begin{cases} (\nabla - iA)^2\Phi + \lambda(1 - |\Phi|^2)\Phi = 0 & \text{in } \Omega, \\ \dfrac{\partial\Phi}{\partial\nu} - i\langle A \cdot \nu\rangle\Phi = 0 & \text{on } \partial\Omega, \\ \operatorname{rot}\operatorname{rot} A + (i(\bar{\Phi}\nabla\Phi - \Phi\nabla\bar{\Phi})/2 + |\Phi|^2 A)\Lambda_\Omega = 0 & \text{in } \boldsymbol{R}^3. \end{cases} \tag{1.2}$$

Here $\langle\cdot,\cdot\rangle$ is the standard inner product of vectors in $\boldsymbol{R}^3$, ν is the unit outward normal vector on $\partial\Omega$ and Λ_Ω is the characteristic function of Ω, i.e. $\Lambda_\Omega(x) = 1$ in Ω and $\Lambda_\Omega(x) = 0$ in $\boldsymbol{R}^3\backslash\Omega$. In [**15**], it was proved for the case of a ring-shaped (rotationally symmetric) domain Ω, that many kinds of stable steady state solutions coexist for large $\lambda > 0$. The purpose of this paper is to extend this result to a general non-trivial domain Ω (cf. Fig. 1). Here the general non-trivial domain means the general domain that is not simply-connected. In [**16**] the GL functional and its variational equation (cf. (1.3), (1.4)) simplified by neglecting the magnetic effect, were studied and several kinds of

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