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The Sylvester's law of inertia in simple graded Lie algebras

Dedicated to Professor Ichiro Satake on the occasion of his seventieth anniversally birthday

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Introduction.

Let $H_n(\boldsymbol{R})$ be the vector space of $n \times n$ real symmetric matrices. The group $GL(n, \boldsymbol{R})^0$ (= the identity component of $GL(n, \boldsymbol{R})$) acts on $H_n(\boldsymbol{R})$ by the rule: $X \mapsto AX\,{}^tA$, $X \in H_n(\boldsymbol{R})$, $A \in GL(n, \boldsymbol{R})^0$. The Sylvester's law of inertia asserts that, by this action of $GL(n, \boldsymbol{R})^0$, X is transformed into the canonical form $\operatorname{diag}(1, \dots, 1, -1, \dots, -1, 0, \dots, 0)$, which is uniquely determined by X. The simple Lie algebra $\mathfrak{sp}(n, \boldsymbol{R})$ has a unique gradation $\mathfrak{sp}(n, \boldsymbol{R}) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, where $\mathfrak{g}_{-1} = H_n(\boldsymbol{R})$ and $\mathfrak{g}_0 \simeq \mathfrak{gl}(n, \boldsymbol{R})$. The $GL(n, \boldsymbol{R})^0$-module $H_n(\boldsymbol{R})$ is imbedded in $\mathfrak{sp}(n, \boldsymbol{R})$ as the G_0^0-module $\mathfrak{g}_{-1}$, where G_0^0 is the analytic subgroup of Aut $\mathfrak{g}$ generated by $\mathfrak{g}_0$. The Sylvester's law of inertia for $H_n(\boldsymbol{R})$ is no other than obtaining the complete representatives of G_0^0-orbits in $\mathfrak{g}_{-1}$. As a generalization of this situation, one can pose:

PROBLEM. Let $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ be a real simple graded Lie algebra, G_0 the group of grade-preserving automorphisms of $\mathfrak{g}$ and let G_0^0 be the identity component of G_0. Find the G_0^0-orbit decomposition and the G_0-orbit decomposition of $\mathfrak{g}_{-1}$.

When $\nu = 1$, this problem is equivalent to the problem of finding the orbits in a compact simple Jordan triple system under the structure group or the identity component of the structure group. Also it is equivalent to finding the orbit decomposition of a tangent space by the linear isotropy group for a symmetric R-space.

The purpose of this paper is to settle the above problem for the case $\nu = 1$ by a unified method. Partial answers have been obtained by Satake [**22, 23**], Kaneyuki [**9, 10**] and Takeuchi [**27**]. In the following we shall describe briefly how to get the two kinds of orbit decompositions of $\mathfrak{g}_{-1}$. The sections 1 and 2 are preliminary sections. We give a quick review for the followings: classification and construction of gradations in semisimple Lie algebras [**13, 12**], the root theory in simple graded Lie algebras $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ ([**13**]), the Jordan triple system $\mathfrak{B}$ on $\mathfrak{g}_{-1}$ (Loos [**18**]) and the root-theoretic version of a frame (= a maximal system of pairwise orthogonal idempotents) $\{e_1, \dots, e_r\}$ in $\mathfrak{g}_{-1}$, and the Jordan algebra structure $\mathfrak{A}_p$ $(0 \le p \le r)$ in $\mathfrak{g}_{-1}$. In §3, applying a result of Matsumoto [**19**], we get a set of good representatives of G_0 mod G_0^0, which allows us to get the G_0-orbit decomposition from the G_0^0-orbit decomposition. We consider the root system Δ^* corresponding to a certain symmetric real flag domain M^*. It turns out that the Weyl group $W(\Delta^*)$ of Δ^*, viewed as a subgroup of G_0^0, acts on the frame $\{e_1, \dots, e_r\}$ as signed permutations. Then we can choose the candidates $o_{p,q}$ $(0 \le p, q \le r, p + q \le r)$ of representatives of the G_0^0-orbits, which are defined in