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Asymptotic behaviour of solutions to non-isothermal phase separation model with constraint in one-dimensional space

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1. Introduction.

Let us consider a one-dimensional model for non-isothermal phase separation, which is given by the following system, denoted by (PSC):

$$[\rho(u)+\lambda(w)]_t - u_{xx} = f(t,x) \quad \text{in } Q := (0,+\infty)\times J, \tag{1.1}$$

$$w_t - \{-\kappa w_{xx} + \xi + g(w) - \lambda'(w)u\}_{xx} = 0 \quad \text{in } Q, \tag{1.2}$$

$$\xi \in \partial I_{[\sigma_*,\sigma^*]}(w) \quad \text{in } Q, \tag{1.3}$$

$$-u_x(t,-L) + n_0 u(t,-L) = h_-(t), \quad u_x(t,L) + n_0 u(t,L) = h_+(t) \quad \text{for } t>0, \tag{1.4}$$

$$w_x(t,-L) = w_x(t,L) = 0 \quad \text{for } t>0, \tag{1.5-1}$$

$$[-\kappa w_{xx}(t,\cdot) + \xi(t,\cdot) + g(w(t,\cdot)) - \lambda'(w(t,\cdot))u(t,\cdot)]_x|_{x=\pm L} = 0 \quad \text{for } t>0, \tag{1.5-2}$$

$$u(0,x) = u_0(x), \quad w(0,x) = w_0(x) \quad \text{for } x \in J. \tag{1.6}$$

Here $J := (-L,L)$ with a positive number L; $k>0$ and $n_0>0$ are constants; $\rho(u)$ is an increasing function of u, and $\lambda(w)$, $\lambda'(w) = (d/dw)\lambda(w)$, $g(w)$ are smooth functions of w; $\partial I_{[\sigma_*,\sigma^*]}$ is the subdifferential of the indicator function $I_{[\sigma_*,\sigma^*]}$ of the interval $[\sigma_*,\sigma^*] \subset R$; $f(t,x)$, $h_\pm(t)$, $u_0(x)$ and $w_0(x)$ are given data.

The above system arises in the phase separation of a binary mixture with components A and B. In this context, $\theta := \rho(u)$ represents the absolute temperature and $w := w_A$ the order parameter which is the local concentration of the component A; note that $\sigma_* = 0 \le w_A(t,x) \le 1 = \sigma^*$, and $w_A(t,x) = 1$ (resp. $w_A(t,x) = 0$) means that the phase (the physical situation of the system) at (t,x) is of pure A (resp. pure B), while $0 < w_A(t,x) < 1$ means that the phase at (t,x) is of mixture. Along the same approach as [**1**, **13**], the system (1.1)–(1.3) can be derived from a free energy functional of Landau-Ginzburg type

$$F_\Omega(\theta;w) := \int_J \left\{ \frac{\kappa\theta}{2}|w_x|^2 + \tau(\theta) + \theta(I_{[0,1]}(w) + \hat{g}(w)) + \lambda(w) \right\} dx \quad \text{for } w \in H^1(J),$$

where $\hat{g}$ is a primitive of g and $\tau(\theta)$ is a smooth function of θ satisfying $\theta = \tau(\theta) - \theta\tau'(\theta)$ $(=\rho(u))$.