On the Cohen-Macaulayness of multi-Rees algebras and Rees algebras of powers of ideals

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1. Introduction.

Let $(A, \mathfrak{m})$ be a Noetherian local ring and I an ideal in A. For a positive integer r, the r-th *multi-Rees algebra* of A with respect to I is defined as the N_0^r-graded ring:

$$R_A(\mathbf{I}_r) = \bigoplus_{\underline{n} \geq \underline{0}} I^{n_1} \cdots I^{n_r}$$

where $\underline{n}$ stands for an r-tuple $(n_1, \ldots, n_r)$ of natural numbers. Very often one identifies $R_A(\mathbf{I}_r)$ with $A[It_1, It_2, \ldots, It_r] \subseteq A[t_1, t_2, \ldots, t_r]$ where $t_1, t_2, \ldots, t_r$ are indeterminates over A. The notation $\mathbf{I}_r$ should be considered as the r-tuple $(I, \ldots, I)$. In particular, in case $r = 1$, the multi-Rees algebra is nothing else but the (ordinary) *Rees algebra* which we denote by $R_A(I)$. For the following, we denote the maximal homogeneous ideal of $R_A(I)$ by M.

The concept of multi-Rees algebras is e.g. connected with mixed multiplicities (cf. [Te] and [HHRTa]) and joint reductions (cf. [Re]). Verma studied for instance the Cohen-Macaulayness of multi-Rees algebras of ideals having joint reduction number zero (see [Ve]). He considered multi-Rees algebras with respect to different ideals $I_1, \ldots, I_r$ of A, i.e. A-algebras of the form: $A[I_1 t_1, I_2 t_2, \ldots, I_r t_r]$, but only in the case that $\dim A = 2$ (plus some additional assumptions). A result on the Cohen-Macaulayness of multi-Rees algebras of $\mathfrak{m}$-primary ideals in a local ring A of dimension two having joint reduction number zero can also be found in [HHRTa]. From a totally different viewpoint, Goto and Nishida studied the Cohen-Macaulay and Gorenstein property of $R_A(\mathbf{I}_2)$ in their work on Rees algebras defined by a filtration ([GN]). A detailed characterization of the Cohen-Macaulay and Gorenstein property of multi-Rees algebras was given by Herrmann, Hyry and Ribbe (et al.) in [HHR1], [HHR2] and [HHRTa]. (Their results do not only deal with properties of $R_A(\mathbf{I}_r)$, but also with properties of multi-Rees algebras of the form $A[I^{k_1} t_1, \ldots, I^{k_r} t_r]$, where $k_1, \ldots, k_r$ are positive integers.) Some newer results on the Cohen-Macaulayness of multi-Rees algebras and its relation with filter-regularity in the corresponding form ring can be found in [HHK].

One of the most useful results on the Cohen-Macaulay property of multi-Rees algebras, which we will also frequently use throughout this paper, is (cf. [HHR1, 2.2]):