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Global weak entropy solutions to quasilinear wave equations of Klein-Gordon and Sine-Gordon type

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1. Introduction.

In this paper we establish the existence of global Lipschitz continuous solutions to the Cauchy problem for the one-dimensional quasilinear wave equation

$$\partial_t^2 w - \partial_x \sigma(\partial_x w) + f(w) = 0, \tag{1.1}$$

for all $(x,t) \in \boldsymbol{R} \times (0,\infty)$, with initial conditions

$$w(x,0) = w_0(x), \quad \partial_t w(x,0) = w_1(x), \tag{1.2}$$

for all $x \in \boldsymbol{R}$. Here f is a smooth function with $f(0) = 0$ and σ is a given smooth function such that $\sigma'(u) \geq \gamma > 0$ $(\gamma > 0)$ and $u\sigma''(u) > 0$ for $u \neq 0$; w_0 and w_1 are bounded functions with compact support, w_0 is also Lipschitz continuous.

This equation models a vibrating string with an elastic external positional force and can also be deduced (at a very formal level) by applying the principle of the "stationary action" from the Lagrangian density given by

$$\mathscr{L}_1(w_t, w_x, w) = \frac{1}{2} w_t^2 - \Sigma(w_x) - F(w)$$

where $\Sigma' = \sigma$ and $F' = f$.

As an example we can consider the quasilinear Klein-Gordon equation

$$\partial_t^2 w - \partial_x \sigma(\partial_x w) + mw = 0 \quad (m \in \boldsymbol{R}) \tag{1.3}$$

and the quasilinear Sine–Gordon equation

$$\partial_t^2 w - \partial_x \sigma(\partial_x w) + \sin w = 0. \tag{1.4}$$

Let us notice that the semilinear versions of the equations (1.3), (1.4) exhibit linear dispersive waves **[Wh]**, although this behaviour has not yet been analyzed in detail in the present case.

The Cauchy problem (1.1)–(1.2) will be considered in the following equivalent formulation. Denote by

$$u = \partial_x w, \quad v = \partial_t w. \tag{1.5}$$

Key words and phrases. Klein-Gordon and Sine-Gordon equations, hyperbolic nonhomogeneous systems, shock waves, fractional step approximation, compensated compactness.