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Integral formulas for polyhedral and spherical billiards

Dedicated to Professor Yoshihiro Tashiro on his 70th birthday

By Nobuhiro INNAMI

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1. Introduction.

Let M^{n+1} be a complete Riemannian manifold with boundary $\partial M =: B \neq \phi$ which is a union of smooth hypersurfaces. We can see the precise definition of manifolds with boundary as billiard tables in [**17**]. Let $q \in B$ be an arbitrary point at which B is smooth and Q_q the symmetry with respect to T_qB, i.e.,

$$Q_q(w) = w - 2\langle w, N(q)\rangle N(q)$$

for any $w \in T_qM$, where $\langle \cdot, \cdot \rangle$ is the Riemannian metric in M and N is the unit normal vector field to B pointing inward. We say that $\gamma : [a,b] \to M$ is a *reflecting geodesic* or briefly a *geodesic* if there exists the partition $a = a_0 < a_1 < \cdots < a_m = b$ such that

(1.1) $\gamma(a_i) \in B$, B is smooth at $\gamma(a_i)$ and $\dot{\gamma}(a_i - 0) \notin T_{\gamma(a_i)}B$ for $i = 1, 2, \ldots, m-1$.

(1.2) $\gamma_i = \gamma | [a_{i-1}, a_i]$ is a geodesic in M in the usual sense for $i = 1, 2, \ldots, m$.

(1.3) $Q(\dot{\gamma}(a_i - 0)) = \dot{\gamma}(a_i + 0)$ for $i = 1, 2, \ldots, m-1$.

Throughout the paper the term "*geodesic*" means both usual one and reflecting one. We assume that geodesics are parametrized by arclength. As usual a variation of a geodesic γ through geodesics yields a Jacobi vector field Y along γ which satisfies the following properties at the boundary (see Section 2):

$$Q(Y(a_i - 0)) = Y(a_i + 0) \tag{1.4}$$

$$Q(\nabla_{\dot{\gamma}(a_i - 0)} Y) - \nabla_{\dot{\gamma}(a_i + 0)} Y = A(\dot{\gamma}(a_i + 0))(Y^{\perp}(a_i + 0)) \tag{1.5}$$

where $A(\dot{\gamma}(a_i + 0))$ is a symmetric endomorphism of n-dimensional subspace $\dot{\gamma}(a_i + 0)^{\perp}$ of $T_{\gamma(a_i)}M$ which is perpendicular to $\dot{\gamma}(a_i + 0)$ and ∇ is the Levi-Civita connection. We say that $\gamma(t_1)$, $t_0 \neq t_1 \in [a,b]$, is a *conjugate point* to $\gamma(t_0)$, $t_0 \in [a,b]$, if there exists a nontrivial Jacobi vector field Y along γ with $Y(t_0) = Y(t_1) = 0$.

Let T_1M be the unit tangent bundle of M. For a $v \in T_1M$ let γ_v be the geodesic with $\dot{\gamma}_v(0) = v$. If $\pi(v) \in B$ where $\pi : T_1M \to M$ is the natural projection, then $\dot{\gamma}_v(0)$ is considered either $\dot{\gamma}_v(+0)$ or $\dot{\gamma}_v(-0)$. The geodesics γ_v are defined on the whole real line $(-\infty, \infty)$ for almost all $v \in T_1M$. We denote the set of all such vectors by SM. Let $f^t : SM \to SM$ be a flow given by $f^tv = \dot{\gamma}_v(t)$ for any $v \in SM$. We denote the set of all