J. Math. Soc. Japan
Vol. 50, No. 2, 1998

Nonlinear small data scattering for the wave equation in R^{4+1}

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(Received Feb. 28, 1996)

1. Introduction.

In this paper we discuss nonlinear small data scattering problem for the following wave equation

$$\Box u = F(u), \quad t \in \boldsymbol{R}, x \in \boldsymbol{R}^n. \tag{1.1}$$

Here $\Box = \partial_t^2 - \Delta = \partial_t^2 - \sum_{j=1}^n \partial_j^2$, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ and $F(u) = \lambda |u|^{p-1} u$ with some $\lambda \in \boldsymbol{R} \backslash \{0\}$, $p > 1$. Although our proof also works for complex-valued solutions, we consider only real-valued solutions for simplicity throughout this paper.

Let us first review the previous works on small data scattering for (1.1). Although we shall deal with 4-space dimensional case only, the large number of papers has been devoted to the study of small data scattering for (1.1) in various spaces of functions and in general space dimension $n \geq 2$. Denote by $\dot{W}^{1,p}(\boldsymbol{R}^n)$ $(1 < p < \infty)$ the completion of test functions with respect to the seminorm $\|\nabla v\|_{L^p}$. $\dot{W}^{-1,p'}(\boldsymbol{R}^n)$ $(1/p + 1/p' = 1)$ means the dual space of $\dot{W}^{1,p}(\boldsymbol{R}^n)$. We simply denote $\dot{W}^{1,2}(\boldsymbol{R}^n)$ by $\dot{H}^1(\boldsymbol{R}^n)$. Note that $\dot{H}^1(\boldsymbol{R}^n)$ is identical to $\{v = v(x) | v \in L^{2n/(n-2)}(\boldsymbol{R}^n), \nabla v \in L^2(\boldsymbol{R}^n)\}$ when $n \geq 3$. Set $E(\boldsymbol{R}^n) = \dot{H}^1(\boldsymbol{R}^n) \times L^2(\boldsymbol{R}^n)$. For more information on the definitions of spaces, norms and operators, see Section 2. There are two fundamental problems in the nonlinear scattering theory. One of them is to prove the existence of the wave operators. Pecher [15] established the space-time mixed norm estimates of free solutions of finite energy and proved the existence of the wave operators for (1.1) as mappings from a neighborhood of 0 in $E(\boldsymbol{R}^n)$ into $E(\boldsymbol{R}^n)$, assuming $p = 1 + 4/(n-2)$ and $n = 3, 4, 5$. Ginibre and Velo [2] have eliminated the restriction of n in the result of Pecher and proved the same result for all $n \geq 3$ by making better use of the space-time integrability and estimating fractional derivatives of the nonlinear term in the Besov spaces (see Proposition 3.3 in [2]). $E(\boldsymbol{R}^n)$ is called the energy space and it is the largest space of data for which we may construct the wave operators for (1.1) in the usual sense. Hence a class of data in the results of Pecher, Ginibre and Velo is the largest, but the allowed value of p is $1 + 4/(n-2)$ only. For a smaller class of data we can discuss the scattering theory for a more general perturbation operator. In fact, Strauss [19] proved that the wave operators can be defined for (1.1) as mappings from a neighborhood of 0 in $(\dot{H}^1 \cap \dot{W}^{1,1+1/p}) \times (L^2 \cap L^{1+1/p})$ into $(\dot{H}^1 \cap L^{p+1}) \times (L^2 \cap \dot{W}^{-1,p+1})$, assuming $p_1(n) < p \leq 1 + 4/(n-1)$, $n \geq 2$. Here $p_1(n) = (n + 2 + \sqrt{n^2 + 8n})/2(n-1)$. Later, Mochizuki and Motai [13] reduced the lower bound $p_1(n)$ to a smaller value $p_2(n)$ $(n \geq 2)$ by working in a different space. The lower bound for p in [13], however, does not seem