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The Cauchy problem for Schrödinger type equations with variable coefficients

Dedicated to Professor Toshinobu Muramatsu on his 60th birthday

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§1. Introduction

In this article we consider the following Cauchy problem in $(0,T)\times \boldsymbol{R}^n$,

$$(1.1)\qquad \begin{aligned} &L[u(t,x)] = f(t,x), \quad (t,x)\in(0,T)\times \boldsymbol{R}^n \\ &u(0,x) = u_0(x), \quad x\in \boldsymbol{R}^n, \end{aligned}$$

where $L[u] = \partial_t u - \sqrt{-1}\sum_{j,k}\partial_j\{a_{jk}(x)\partial_k u\} - \sum_j b_j(t,x)\partial_j u - c(t,x)u$ and $\partial_t = \partial/\partial t$ and $\partial_j = \partial/\partial x_j$. We assume that $a_{jk}(x)$ belong to B^∞ and $b_j(t,x), c(t,x)$ are in $C^0([0,T];B^\infty)$, where B^∞ stands for the set of complex valued functions defined in $\boldsymbol{R}^n$ whose all derivatives are bounded in $\boldsymbol{R}^n$. For a topological space X, a non negative integer k and an interval I in R^1 we denote by $C^k(I;X)$ the set of functions k times continuously differentiable with respect to $t\in I$ in the topology of X. Moreover we assume that $a_{jk}(x) = a_{kj}(x)$ are real valued and there is $c_0>0$ such that

$$(1.2)\qquad \sum_{j,k} a_{jk}(x)\xi_j\xi_k \ge c_0|\xi|^2, \quad x,\xi\in R^n.$$

Let $T>0$ and X a topological space. We say that the Cauchy problem (1.1) is X-well posed in $(0,T)$, if for any u_0 in X and any f in $C^0([0,T];X)$ there exists a unique solution u in $C^0([0,T];X)$ of (1,1).

We shall prove that the Cauchy problem (1.1) is X-well posed in $(0,T)$ under some assumptions, if we take $X = L^2(\boldsymbol{R}^n)$ the set of square integrable functions in $\boldsymbol{R}^n$ or $X = H^\infty$ the sobolev space in $\boldsymbol{R}^n$.

We know a necessary condition in order that the Cauchy problem is L^2 (resp. H^∞)-well posed in $(0,T)$. To state this we need the classical orbit associated to L. Put

$$(1.3)\qquad a_2(x,\xi) = \sum_{j,k} a_{jk}(x)\xi_j\xi_k$$

and let $(X(t,y,\eta),\Xi(t,y,\eta))$ be the solution of the following ordinary differential equations

$$(1.4)\qquad \begin{aligned} &(d/dt)X_j(t) = (\partial/\xi_j)a_2(X(t),\Xi(t)), \quad X_j(0) = y_j \\ &(d/dt)\Xi_j(t) = -(\partial/\partial x_j)a_2(X(t),\Xi(t)), \quad \Xi_j(0) = \eta_j, \end{aligned}$$