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A topology on the semigroup of endomorphisms on a von Neumann algebra

Dedicated to Professor Richard V. Kadison on the occasion of his 70th birthday.

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1. Introduction.

1.1. The automorphism group of a von Neumann algebra reflects the structure of the algebra in many ways, and this idea was a cornerstone in the first pioneering works of A. Connes (cf. e.g. [1]). In fact the classification of (groups of) automorphisms is essential to most known classification theorems in von Neumann algebra. A main reason for the usefulness of this approach is that the automorphism group $\mathrm{Aut}(M)$ of a von Neumann algebra M can be topologized in a very nice way. Of course, several natural locally convex topologies on $\mathrm{Aut}(M)$ can be defined, but one of them is particularly useful, namely the u-topology (cf. [4]). A net (α_n) in $\mathrm{Aut}(M)$ converges to $\alpha \in \mathrm{Aut}(M)$ in the u-topology if and only if

$$\varphi \circ \alpha_n \longrightarrow \varphi \circ \alpha, \quad \varphi \in M_*,$$

and we then write $\alpha_n \xrightarrow{u} \alpha$. One nice property is that, in this topology, $\mathrm{Aut}(M)$ is a Polish topological group, in fact in a standard representation $M \subseteq \mathscr{B}(H)$ there is a multiplicative homeomorphism of $\mathrm{Aut}(M)$ onto a closed subgroup of the unitary group of H [4].

1.2. Motivated by striking applications in mathematical physics, as well as by quite surprising connections with mathematical disciplines such as knot theory and quantum Lie groups, the classification of *inclusions* of von Neumann algebras has become an object of intense study over the last 12 years. As an attempt to mimic the automorphism approach for single factors, given an inclusion $M \supseteq N$, one can study the group $\mathrm{Aut}(M,N)$ of automorphisms of M that preserve N globally. This has been quite successful in an indirect way, but of course we do not get any direct information on the subalgebras of M since we actually fixed N. In terms of endomorphisms on M, this corresponds to limit attention to those endomorphisms that map a fixed subalgebra into itself, or, even more specialized, onto the subalgebra. We shall denote the semigroup of endomorphisms of M by $\mathrm{End}(M)$, and by $\mathrm{End}(M,N)$ we mean the elements of

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