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Asymptotic behavior of least energy solutions to a semilinear Dirichlet problem near the critical exponent

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1. Introduction

Let Ω be a smooth bounded domain in R^n with $n \geq 3$ and $p = (n+2)/(n-2)$ (the Sobolev exponent). Consider the problem

$$(1.1) \qquad \begin{cases} -\Delta u = u^{p-\varepsilon} \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\varepsilon > 0$. It is well-known that when $\varepsilon > 0$, problem (1.1) has at least one solution. On the other hand, when $\varepsilon = 0$, problem (1.1) becomes delicate. Pohozaev [**12**] derived the so-called "Pohozaev identity" for (1.1) and showed the nonexistence of solutions to (1.1) when Ω is star-shaped. In other cases, Bahri and Coron [**2**] showed that there exists a solution for equation (1.1) when Ω has a nontrivial topology, while Ding [**D**] constructed a solution to (1.1) when Ω is contractible. Here arises an interesting question: what happens to the solutions of (1.1) as $\varepsilon \to 0$? The first result was due to Atkinson and Peletier in [**1**]. They studied the radial case and characterized the asymptotic behavior of radial solutions. Later, Brezis and Peletier [**3**] used PDE methods to give another proof of the same result in spherical domains. Finally, Z. Han [**9**] (independently by O. Rey [**13**]) proved the same result in the general case, namely:

THEOREM A. *Let u_ε be a solution of problem* (1.1) *and assume*

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2}{\|u_\varepsilon\|^2_{L^{p+1-\varepsilon}(\Omega)}} = S + o(1) \quad \text{as } \varepsilon \to 0,$$

where S is the best Sobolev constant in $R^n : S = \pi n(n-2)(\Gamma(n/2)^{n/2}/\Gamma(n))$. Suppose u_ε assumes its maximum at x_ε. Then we have (after passing to a subsequence):

1. *There exists $x_0 \in \Omega$ such that as $\varepsilon \to 0$, $x_\varepsilon \to x_0$, $u_\varepsilon \to 0$ in $C^1_{\mathrm{loc}}(\bar{\Omega}\backslash\{x_0\})$ and $|\nabla u_\varepsilon|^2 \to (n(n-2))^{-(n-2)/4}\delta_{x_0}$ in the sense of distribution, where δ_{x_0} is the Dirac function at point x_0.*