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# A functional decomposition theorem for the conformal representation

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## 1. Introduction.

This paper concerns the nonlinear operator which maps an injective function $\phi$ defined on the unit disk in $\boldsymbol{R}^2$ with values in $\boldsymbol{R}^2$, of class $C^{m,\alpha}$, and with nonvanishing jacobian, into the unique holomorphic, one-to-one map $g[\phi]$ of the unit disk $\mathscr{D}$ onto the Jordan domain $\phi(\mathscr{D})$, normalized by $g[\phi](0)=\phi(0)$, $g'[\phi](0)\in]0, +\infty[$. By virtue of Warschawski's work, it is easy to see that under our assumptions $g[\phi]\in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ if $\alpha\in]0, 1[$. We have chosen to represent the Jordan domain as the image of $\phi$, rather than as the more customary Jordan curve which parametrizes the boundary of the Jordan domain to allow the application of the methods of this paper, but as we show in section 2, there is no loss of generality.

The main finding of this paper is that the nonlinear operator $\phi\mapsto g[\phi]$ can be decomposed as $\phi\mapsto\phi\circ S[\phi]^{(-1)}$, where $S[\phi]^{(-1)}$ denotes the inverse function of $S[\phi]$, and the operator $\phi\mapsto S[\phi]$ is analytic from a set of 'admissible' $\phi$'s in $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ to $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. In other words, $\phi\mapsto g[\phi]^{(-1)}\circ\phi$ is analytic. As we shall explain, this result easily allows to give precise information on the regularity of $\phi\mapsto g[\phi]$.

The analyticity statement for $S[\cdot]$ may sound surprizing. Indeed, in section 5 we show that $\phi\mapsto g[\phi]$ is not even differentiable from the set of admissible $\phi$'s in $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ to $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$, and we show that in order to have differentiability of $g[\cdot]$, we must increase the regularity of the elements in the domain of $g[\cdot]$. We note that the problem of expanding $g$ in a power series of a parameter $\varepsilon$ ranging on some interval $I$ of the real line, when the Jordan domain depends on $\varepsilon$ and is for each $\varepsilon$ parametrized by the elements of $\partial\mathscr{D}$, has been solved by Kantorovich in the thirties (cf. Kantorovich & Krylov

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