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Del Pezzo surfaces as hyperplane sections

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Introduction.

Let L be a very ample line bundle on an n-dimensional complex projective manifold X. In this article we classify pairs (X, L) as above with some smooth $A\in|L|$ being del Pezzo, i.e., with a smooth $A\in|L|$ such that $-K_A=(n-2)H$ for some ample line bundle H on A. We assume that $n\geqq 3$ since otherwise we are in the completely understood case when A is an elliptic curve.

If $H=L_A$, then the problem reduces to the classification of del Pezzo manifolds, which has been done by Fujita [**Fu**] in the more general setting of ample divisors. However there are several examples (e.g., [**LPS**], [**LPS1**]) showing that $H\neq L_A$ can occur. This suggests the development of a detailed structure theory in which both Fujita's theory (in the very ample setting) and all known examples fit. This is exactly what we do in this paper.

If $n\geqq 4$, then the structure of pairs (X, L) with $A\in|L|$ del Pezzo is simple: we work it out in the appendix. In particular this shows that, apart from few obvious exceptions, the situation $H\neq L_A$ can occur only when $n=3$, which we assume from here on in this introduction.

In section 0 we summarize background material. We also prove some very ampleness results (Theorems (0.3) and (0.5)) in order to show that a number of pairs coming up in the classification do really occur.

In section 2, by using adjunction theory, we prove a structure theorem (Theorem (2.4)) giving a breakup of the possible pairs (X, L) we are dealing with into 9 classes. Of these the most complicated are quadric fibrations over $\boldsymbol{P}^1$, Veronese bundles over $\boldsymbol{P}^1$ and scrolls over surfaces.

We study quadric fibrations over $\boldsymbol{P}^1$ in sections 1 and 4. To do this we embed X in $\boldsymbol{P}(\pi_*L)$ where $\pi: X\rightarrow\boldsymbol{P}^1$ is the quadric fibration map. The smoothness of X imposes very strong restrictions on which vector bundles π_*L are possible and on the homology class of X in $\boldsymbol{P}(\pi_*L)$.

In section 3 we classify the Veronese bundles over $\boldsymbol{P}^1$ that arise in our structure theory.