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On compact Kähler-Liouville surfaces

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Introduction.

It is rewarding to investigate riemannian manifolds with completely integrable geodesic flows, because the behavior of their geodesics can be observed.

In the 19th century, Jacobi investigated the 2-dimensional ellipsoid, and Liouville generalized this work to the geometry of a class of metrics, the so-called Liouville line elements, whose geodesic flows are integrable by a certain first integral. In relation to the present viewpoint, their investigations can be recognized as a local theory of differential geometry. However, in 1991 K. Kiyohara began to develop a global theory in this area [**1**]. In this work he first defined the compact Liouville surface and classified it; it is defined as a compact 2-dimensional riemannian manifold whose geodesic flow has a first integral on the cotangent bundle such that (1) the first integral is fiberwise a homogeneous polynomial of degree 2; (2) the first integral can not be expressed as a linear combination of the square of a certain vector field and its energy function. Additionally, K. Sugahara, K. Kiyohara and the author investigated noncompact Liouville surfaces [**2**]. Subsequently, Kiyohara generalized this concept to the higher dimensional manifolds (see [**3**] for detail) as follows:

A Liouville manifold is defined as a riemannian manifold which has a real vector space of the first integrals on the cotangent bundle of its geodesic flows such that (1) all the first integrals are fiberwise homogeneous polynomials of degree 2; (2) all the first integrals are simultaneously normalizable on each fiber; (3) the dimension of the vector space is equal to the dimension of the underlying riemannian manifold.

In the investigation [**3**] of Liouville manifolds, Kiyohara has assumed the condition of "properness," and has classified proper Liouville manifolds of rank one; he has concluded that a proper 4-dimensional real Liouville manifold of rank one is diffeomorphic with the sphere S^4, the real projective space $\boldsymbol{RP}^4$ or the euclidean space $\boldsymbol{R}^4$.

It is known that the geodesic flow of the n-dimensional complex projective space $\boldsymbol{CP}^n$ $(n \geqq 1)$ equipped with the standard metric is completely integrable (cf. [**4**], [**5**]). The author was informed by private communication with Prof. K. Kiyohara that there is a family of Kähler metrics on $\boldsymbol{CP}^n$ whose geodesic