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Commuting squares and a new relative entropy

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1. Introduction.

A. Connes and E. Størmer [3] defined the relative entropy $H(M|N)$ for finite dimensional von Neumann subalgebras M and N of a finite von Neumann algebra L by using the notion of Umegaki relative entropy. When L is commutative, M and N are generated by some finite partitions P and Q, then $H(M|N)$ coincides with the classical conditional entropy $h(P, Q)$ ([2]). Later, M. Pimsner and S. Popa [13] investigated the relation between the relative entropy $H(M|N)$ and the Jones index $[M:N]$, where N is a subfactor of a factor M of type II_1. Very recently Y. Watatani and J. Wierzbicki [20] computed the relative entropy $H(M|N)$ for two subfactors M and N of a factor of type II_1 without assuming $N\subset M$, which extended the classical formula $h(P, Q)=h(P\vee Q, Q)$ in ergodic theory to the non-commutative case. They showed that the commuting square condition implies $H(M|N)=H(M|M\cap N)$ and the commuting square condition for commutants implies $H(M|N)=H(M\vee N|N)$.

Now, J.I. Fujii and E. Kamei [5] introduced the relative operator entropy $s(a|b)$ for positive operators a, b as a relative version of the Nakamura-Umegaki operator entropy. In the case where a, b are commutative, this relative operator entropy coincides with the Umegaki relative entropy, but in general they do not coincide. On the other hand Belavkin and Staszewski had defined in [1] a relative entropy s_{BS} in C^*-algebra setting. F. Hiai and D. Petz [10] pointed out that $s_{BS}(a, b)\geqq -Tr(s(a|b))$ for density matrices a, b where Tr denotes the usual trace matrices. In noncommutative probability theory, F. Hiai investigated the relation between the Umegaki relative entropy and Belavkin and Staszewski relative entropy and showed some remarkable results in [9], [10].

In the previous paper [15, 16], we introduced an entropy $S(M|N)$ of a finite von Neumann algebra M relative to its subalgebra N as a noncommutative version of the Umegaki relative entropy which is not identical with the Connes-Størmer relative entropy $H(M|N)$ and showed a version of the Pimsner-Popa Theorem on the relative entropy and the Jones index for the factors of type II_1.

In this paper we shall compute the relative entropy $S(M|N)$ without assuming $N\subset M$ and investigate the difference between the relative entropy