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The tightness about sequential fans and combinatorial properties

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1. Introduction.

Let κ be an infinite cardinal. The *sequential fan* S_κ *with* κ*-many spines* is the quotient space obtained from the disjoint union of κ-many convergent sequences by identifying all the limit points to a single point denoted by ∞. To be precise, $S_\kappa=\{\infty\}\cup(\kappa\times\omega)$ as a set, every point of $\kappa\times\omega$ is isolated, and a basic neighborhood of ∞ is of the form

$$U_\varphi=\{\infty\}\cup\{\langle\alpha, n\rangle : n\geqq\varphi(\alpha)\}$$

where $\varphi\in\omega^\kappa$.

For a topological space X, the *tightness* of X, $t(X)$, is the smallest cardinal λ such that for every point $x\in X$ and $A\subseteq X$, if $x\in \mathrm{cl}A$ then there exists $B\subseteq A$ with $|B|\leqq\lambda$ and $x\in \mathrm{cl}B$.

It follows immediately from the definition that $t(X)\leqq|X|$ and it is easily seen that $t(S_\kappa)=\omega$ for each κ. But the tightness of the product space of two sequential fans is more complicated.

Gruenhage [4] proved that $t(S_{\omega_1}\times S_{\omega_1})=\omega_1$, but it is an open question whether $t(S_{\omega_2}\times S_{\omega_2})=\omega_2$ holds in ZFC. Moreover, such a question whether $t(S_\kappa\times S_\kappa)=\kappa$ or not, is equivalent to another question related to the collectionwise Hausdorff property. (See [3, 8] for details.)

In this paper we shall give a combinatorial characterization of the tightness of $S_\omega\times S_\kappa$ for an infinite cardinal κ. Especially the tightness of $S_\omega\times S_{2^\omega}$ has a natural combinatorial characterization.

To begin with, let us review the definitions of two familiar cardinals with combinatorial characterizations, $\mathfrak{b}$ and $\mathfrak{d}$.

DEFINITION 1.1. For $f, g\in\omega^\omega$, $f\leqq^* g$ if for all but finitely many $n\in\omega$ we have $f(n)\leqq g(n)$. A family $\mathcal{F}\subseteq\omega^\omega$ is *unbounded* (respectively *dominating*) if for every $f\in\omega^\omega$ there exists $g\in\mathcal{F}$ such that $g\not\leqq^* f$ (respectively $f\leqq^* g$). The *unbounding number* $\mathfrak{b}$ is the smallest size of the unbounded family of ω^ω, and the *dominating number* $\mathfrak{d}$ is the smallest size of the dominating family of ω^ω.