Graph decompositions without isolated vertices II

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1. Introduction.

All graphs considered in this paper are finite, undirected and without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of $V(G)$, the neighbourhood of v in G, denoted by $N_G(v)$, is the set of vertices of G adjacent to v, and the degree $d_G(v)$ of v in G is $|N_G(v)|$. We denote by $\delta(G)$ and $\kappa(G)$ the minimum degree and the connectivity of G, respectively. For a subset S of $V(G)$, let $\langle S \rangle_G$ denote the subgraph of G induced by S. For standard terms or notation not defined here, see [1] or [2].

Given a graph G of order n and a partition $n = \sum_{i=1}^{k} a_i$ with $a_i \geq 1$, S. B. Maurer [10] conjectured that if $\kappa(G) \geq k$, then $V(G)$ can be decomposed as $V(G) = \bigcup_{i=1}^{k} A_i$ with the conditions $|A_i| = a_i$ and $\kappa(\langle A_i \rangle_G) > 0$ (i.e., $\langle A_i \rangle_G$ is connected) for all i, $1 \leq i \leq k$. A. Frank [7], on the other hand, conjectured the following stronger form of this, which was settled independently by L. Lovász [9] and E. Győri [8].

THEOREM A [9, 8]. *Let G be a graph of order n, and $n = \sum_{i=1}^{k} a_i$ be a partition of n with $a_i \geq 1$. Suppose that $\kappa(G) \geq k$. Then for any distinct k vertices $v_1, \cdots, v_k$ of $V(G)$, $V(G)$ can be decomposed as $V(G) = \bigcup_{i=1}^{k} A_i$ with the conditions $|A_i| = a_i$, $v_i \in A_i$ and $\kappa(\langle A_i \rangle_G) > 0$ for all i, $1 \leq i \leq k$.*

Turning his attention from "connectedness" to "no isolation", Frank also conjectured the following as an analogue of Maurer's conjecture, in which the conditions on the connectivity are replaced by those on the minimum degree. (Note that $\delta(\langle A_i \rangle_G) > 0$ implies that $\langle A_i \rangle_G$ contains no isolated vertices.) Thereafter some partial results on this came out in a row, while a complete proof was finally given by H. Enomoto [4].

THEOREM B [4]. *Let G be a connected graph of order n, and $n = \sum_{i=1}^{k} a_i$ be a partition of n with $a_i \geq 2$. Suppose that $\delta(G) \geq k$. Then $V(G)$ can be decomposed as $V(G) = \bigcup_{i=1}^{k} A_i$ with the conditions $|A_i| = a_i$ and $\delta(\langle A_i \rangle_G) > 0$ for all i, $1 \leq i \leq k$.*